THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR ....G. Robert Graf........

TITLE OF THESIS ....On a Mathematical Theory of Inhomogeneous Isotropic Materials........

DEGREE FOR WHICH THESIS WAS PRESENTED ....M.Sc. (Mathematics)........

YEAR THIS DEGREE GRANTED ....1973........

THE UNIVERSITY OF ALBERTA

ON A MATHEMATICAL THEORY OF INHOMOGENEOUS ISOTROPIC MATERIALS

by

© G. ROBERT GRAF

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1973

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ON A MATHEMATICAL THEORY OF INHOMOGENEOUS ISOTROPIC MATERIALS submitted by G. ROBERT GRAF in partial fulfillment of the requirements for the degree of Master of Science.

ACKNOWLEDGMENT

I would like to express my deep gratitude to my supervisor, Dr. R.J. Tait, for the help, support, and advice he has given me in all my studies here, and in particular for his advice and criticism in the preparation of this thesis.

I would also like to thank Miss Olwyn Buckland for typing the final draft of this thesis.

TABLE OF CONTENTS

INTRODUCTION

The modern theory of continuum mechanics has expanded greatly in the last few years. As interest in finite deformation theory increased, mathematical fundamentals for general types of materials were established, and the techniques of differential geometry were brought to bear on the problems.

In this thesis, we outline some of the general theories, and give a specific application. We generalize a problem of Vaughan's [5], and discuss the deformation of a compressed, laminated cylinder under its own weight.

In the first part, we rely on the concepts introduced by Wang [6], [7] to give a mathematical description of laminated bodies. We develop the theory in terms of local coordinates, and confine our discussion to simple, elastic, materially uniform, isotropic bodies.

In the second part, we use the techniques of Green and Zerna [2] to deal with the problem of small deformations on large. We develop an appropriate notation for the case of laminated or pre-stressed bodies. After deriving the general equilibrium equations for a perturbation, we show how to apply the theory to the stress-system of a universal solution. We choose a particular constitutive equation and a particular metric, to simplify the resulting equations.

Finally, we use numerical techniques to obtain solutions to the equations, subject to appropriate boundary conditions. We show that laminations have a considerable effect on the deformation.

The summation convention is used throughout this thesis, unless specified otherwise. Although we have tried to use a consistent notation, the familiar notation for classical quantities has been retained.

LIST OF SYMBOLS

a,b,u,v - elements of the isotropy group.

A^i_j - the values of the coordinates X^i_j of a point a in G .

B - the left-Cauchy-Green tensor.

B_o - the left-Cauchy-Green tensor of the lamination.

$B_c = FBF^T$

$\tilde{B}$ - an adjusted form of B^2 , defined by (2.2.35).

B_r - one of the vectors spanning $H_{\hat{u}}$.

$\mathcal{B}$ - the physical body.

C - the right-Cauchy-Green tensor.

$e_p, \hat{e}_p, \hat{u}$ - linear frames at p in M .

$e_\alpha(p)$ - the linear frame corresponding to the standard frame of R^3 .

$E^p_q(a)$ - a base vector at a in G .

$\bar{E}^p_q(\hat{a})$ - a base vector at $\hat{a}$ in $E(M)$.

$E(M)$ or $E(\mathcal{B})$ - the bundle of linear frames on M .

$E(M,U)$ or $E(U)$ - the bundle of reference frames.

$E_{\hat{u}}(\mathcal{B})$ or $T_{\hat{u}}(E(\mathcal{B}))$ - the tangent space of $E(\mathcal{B})$ at $\hat{u}$.

$H_{\hat{u}}$ - the horizontal subspace of $E_{\hat{u}}(\mathcal{B})$.

$\mathbf{I}$ - the identity tensor.

I_1, I_2, I_3 - the three principal invariants of B .

I_1 - in Chapter 3, the modified Bessel function.

L - the cube of the compression ratio.

L_β - one of the laminae.

M - a lamination parameter defined by (3.6.1) and (3.6.3).

M_1 - the lamination parameter for the first harmonic.

M - the body manifold.

M_p or $T_p(M)$ - the tangent space of M at p .

p,q - points in M .

$r_p, r_\alpha(p)$ - local configurations of M_p .

r_{pq} - the material isomorphism between M_p and M_q .

R^3 - Euclidean 3-space.

s - a radial variable defined by (3.4.3).

s_1 - the boundary value of s for the first harmonic.

SL(3) - the special linear group for R^3 .

SO(3) - the special orthogonal group for R^3 .

t - the stress vector.

T - a function of the radial variable, defined by (3.2.1).

$T(M)$ - the tangent bundle on M.

$T(E(B))$ - the tangent bundle on $E(B)$.

$\underline{u}_j$ - a base vector for M_p at $\hat{u}$.

(u_α, r_α) - a reference chart on M.

(u_α, η_α) - a material chart on $T(M)$.

$\mathcal{U}$ - a reference atlas for M.

V - a scalar multiple of the function T.

$V_{\hat{u}}$ - the vertical subspace of $E_{\hat{u}}(B)$.

w^β, u, v - the components of the perturbed displacement vector.

W - the strain-energy function.

$\overline{x}, \overline{y}$ - vector fields on $E(M)$.

X - a vector field on G.

X, Y, x, X_o - Cartesian coordinate systems.

Z - a function of the vertical variable, defined by (3.2.1).

ϵ - a perturbation parameter.

ϵ - a lamination parameter defined by (3.6.1).

$\eta_\alpha, \eta_{\alpha p}$ - bundle maps for $T(M)$.

$\eta(U)$ - the material atlas.

$\{\bar{\theta}^\alpha\}, \{\bar{\bar{\theta}}^\alpha\}$ - curvilinear coordinate systems.

$\{\theta^\alpha\}$ - convected coordinate system.

θ, θ' - atlases for $T(M)$.

$\lambda^{\alpha\beta}$ - components of the "perturbed-stress" tensor, defined by (2.2.15).

ξ_α - a bundle map for $E(M)$.

ρ - density.

σ - f divided by the initial height of the cylinder.

σ_1 - the boundary value of σ .

τ - the Cauchy stress tensor.

ϕ, ψ - diffeomorphisms of M to R^3 .

Φ, Ψ - Mooney-Rivlin coefficients.

Special Symbols

$\tilde{z}|_p$ - the natural lift of $z|_p$ to the frame $\hat{u}$.

$\bar{\bar{z}}|_p$ - the special lift of $z|_p$ as defined by (1.5.49).

$(L_z y)|_p$ or $[z,y]|_p$ - the Lie derivative of y with respect to z .

$\nabla_z y$ - the covariant derivative of y with respect to z .

$\bar{Q} = Q + \epsilon\hat{\hat{Q}}$ - the perturbed general quantity Q .

CHAPTER 1

The Material Geometry of Stress

1.1 Materially Uniform Simple Bodies.

The points of a physical body $\mathcal{B}$ will be regarded as points of a closed, connected, 3-dimensional, differentiable, orientable manifold M. M can be covered by one coordinate neighbourhood, i.e. there exist diffeomorphisms

$$\phi : M \to R^3 \tag{1.1.1}$$

where R^3 is Euclidean 3-space with standard orientation.

The tangent space at $p \in M$ is denoted by M_p or $T_p(M)$, where $T(M)$ is the usual tangent bundle. A map

$$r_p : M_p \to R^3 \tag{1.1.2}$$

is called a local configuration if it is an orientation preserving isomorphism. Each diffeomorphism (1.1.1) induces a map at p :

$$\phi_*\big|_p : M_p \to R^3 \quad . \tag{1.1.3}$$

ϕ is called a configuration if $\phi_*\big|_p$ is a local configuration for all p. There need not be any (global) configuration $\psi : M \to R^3$ such that $\psi_*\big|_p \equiv r_p$ for a particular r_p.

We refer to the manifold as M or $\mathcal{B}$ if no confusion will arise.

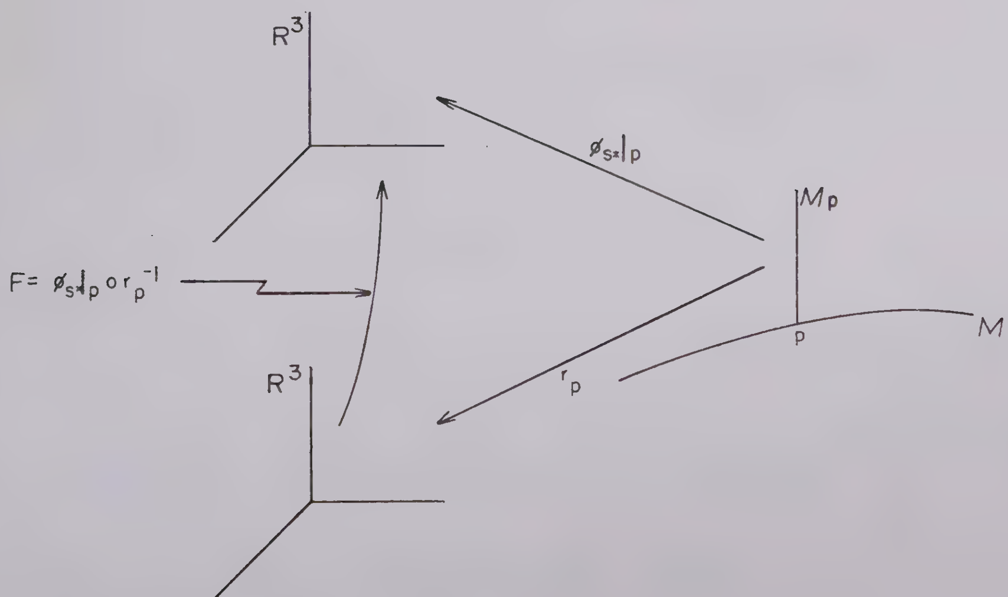
A <u>motion</u> of $\mathcal{B}$ is a map $\Phi : R \times M \to R^4$ defined by

$$\Phi(s,M) = (s,\phi_s(M)) \quad , \tag{1.1.4}$$

where $\phi_s(M)$ is a configuration $\forall$ s . We represent ϕ_s by its <u>deformation gradient</u>

$$F_s : R^3 \to R^3 \quad , \quad F_s = \phi_{s*}\big|_p \circ r_p^{-1} \quad , \tag{1.1.5}$$

where r_p is some local reference configuration. Each choice of r_p gives a different representation of F . If ϕ^i are the coordinates under ϕ_s and $\frac{\partial}{\partial r^j}\Big|_p$ are the base vectors under r_p , then $((F^i_j)) = ((\frac{\partial \phi^i}{\partial r^j}\Big|_p))$ by the chain rule. Since r_p and ϕ_s are orientation preserving, det F > 0 . We have the following diagram.



We define a second-order symmetric stress tensor τ by

$$\tau \underline{n} = \underline{t} \quad , \tag{1.1.6}$$

where $\underline{n}$ is the normal to a surface at a point p of $\mathcal{B}$, and $\underline{t}$ is the stress vector at that point.

$\mathcal{B}$ is called a simple body if τ is determined solely by the local history:

$$\tau(t,p) = \mathop{\mathcal{L}}_{s=-\infty}^{t} (\phi_{s*}|_p, p) \quad , \tag{1.1.7}$$

where t,s denote time and $\mathcal{L}$ is the response functional. For any choice of reference configuration r_p at p, we can refer to the deformation gradient and write

$$\tau(t,p) = \mathop{\mathcal{F}}_{s=-\infty}^{t} (F_s, p) \quad . \tag{1.1.8}$$

If the body is elastic, there is no dependence on the history of the motion, so we may write

$$\tau(t,p) = \tau(p) = \mathcal{R}_p(F) \quad . \tag{1.1.9}$$

We now develop a geometry for $\mathcal{B}$ which is characteristic of its response to stress. We say that two points p,q are materially isomorphic if there exist local configurations r_p, r_q such that

$$\mathcal{R}_p(F) = \mathcal{R}_q(F) \quad \text{for all } F \quad , \tag{1.1.10}$$

i.e. $$R_p(\phi_*|_p \circ r_p^{-1}) = R_q(\phi_*|_q \circ r_q^{-1}) \quad . \tag{1.1.11}$$

If r_p, r_q exist, we have a map

$$r_{pq} : M_p \to M_q \quad , \quad r_{pq} \equiv r_q^{-1} \circ r_p \quad . \tag{1.1.12}$$

We say r_{pq} is a <u>material isomorphism</u>, and we have

$$L_p(\phi_*|_q \circ r_{pq}) \equiv L_q(\phi_*|_q) \quad , \tag{1.1.13}$$

since

$$F = \phi_*|_q \circ r_q^{-1} = \phi_*|_q \circ r_{pq} \circ r_p^{-1} \quad . \tag{1.1.14}$$

We take $\mathcal{B}$ to be a simple, <u>materially uniform</u> body, where by materially uniform we mean that any two points p, q of M are materially isomorphic.

A set $\mathcal{U} = \{(u_\alpha, r_\alpha) : \alpha \in A\}$ is a <u>reference atlas</u> for M if it has the following properties:

1. $\{u_\alpha : \alpha \in A\}$ is an open cover of M.
2. $\{r_\alpha : \alpha \in A\}$ is a smooth field of reference configurations,

$$r_\alpha(p) : M_p \to R^3 \quad \text{for} \quad p \in u_\alpha \quad .$$

3. if $p,q \in u_\alpha$, then $r_\alpha^{-1}(q) \circ r_\alpha(p)$ is a material isomorphism.
4. if $p \in u_\alpha$, $q \in u_\beta$, then $r_\beta^{-1}(q) \circ r_\alpha(p)$ is a material isomorphism.
5. $\mathcal{U}$ is maximal.

On the open set u_α we may write

$$\tau(u_\alpha) = \mathcal{R}_{u_\alpha}(F) \quad , \tag{1.1.15}$$

since by property (3) there is no dependence on the individual points. In fact, by property (4), the charts are compatible on the overlaps, i.e.

$$\mathcal{R}_{u_\beta}(F) = \mathcal{R}_{u_\alpha}(F) \quad \text{on} \quad u_\alpha \cap u_\beta \quad , \tag{1.1.16}$$

so we may write

$$\tau = \mathcal{R}_{\mathcal{U}}(F) \quad . \tag{1.1.17}$$

We deal exclusively with smooth, simple, materially uniform bodies, which may be equipped with a reference atlas.

1.2 The Isotropy Group.

The isotropy group $\mathcal{g}_p$ is the group of all material isomorphisms of p with itself, i.e.

$$\mathcal{g}_p = \{h : \mathcal{B}_p \to \mathcal{B}_p \mid \mathcal{L}_p(\phi_*|_p) \equiv \mathcal{L}_p(\phi_*|_p \circ h)\} \quad . \tag{1.2.1}$$

In addition, we ask that $\mathcal{g}_p \subset SL(\mathcal{B}_p) \ni \det h = +1$.

Rather than deal directly with $\mathcal{B}_p$, we can transfer this concept to R^3 . We say $G : R^3 \to R^3$ is a material isomorphism relative to r_p if

$$\mathcal{R}_p(F) = \mathcal{R}_p(FG) \quad . \tag{1.2.2}$$

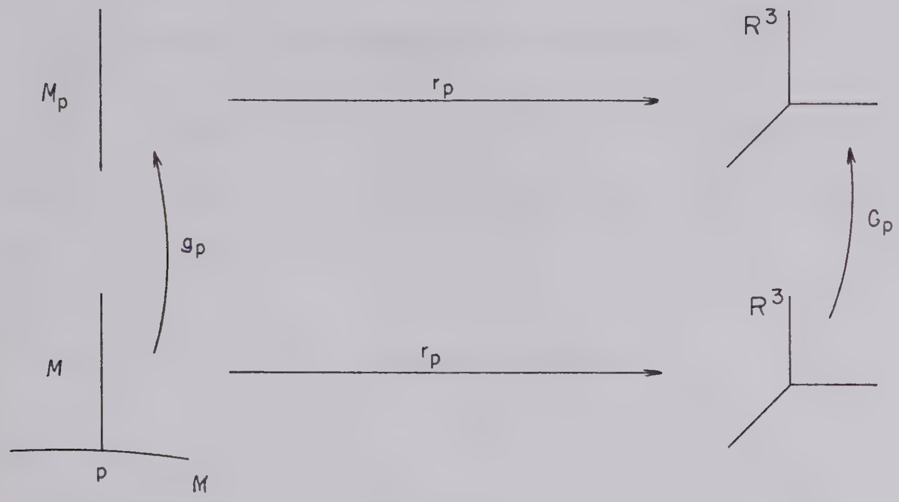


Fig. 1.2.1.

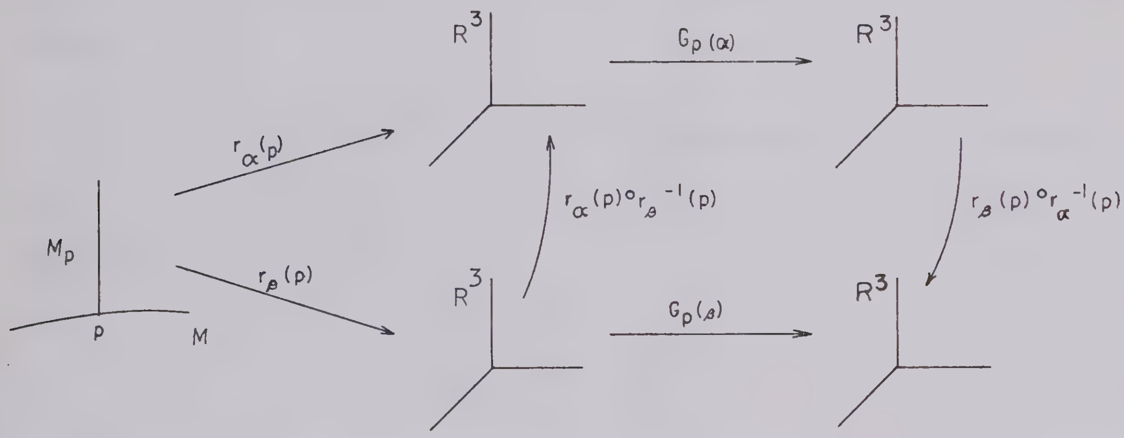


Fig. 1.2.2.

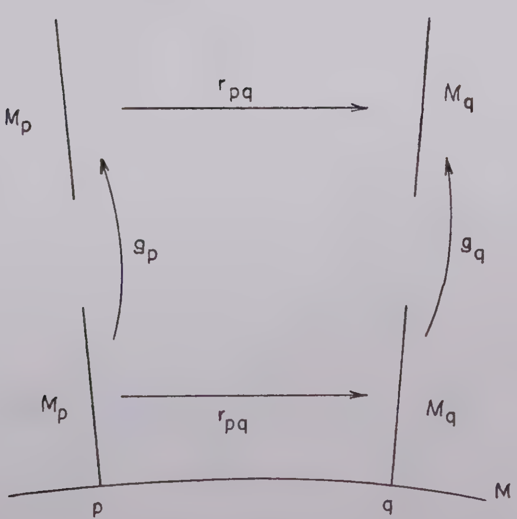


Fig. 1.2.3.

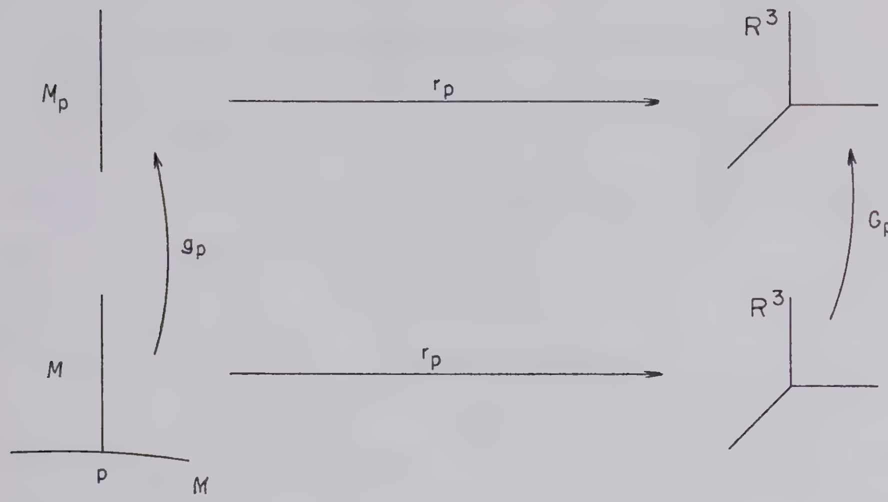


Fig. 1.2.1.

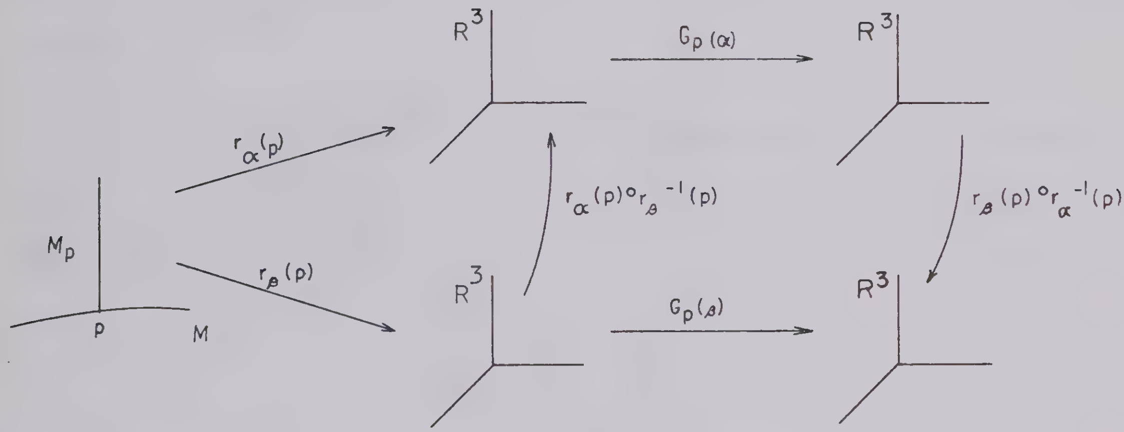


Fig. 1.2.2.

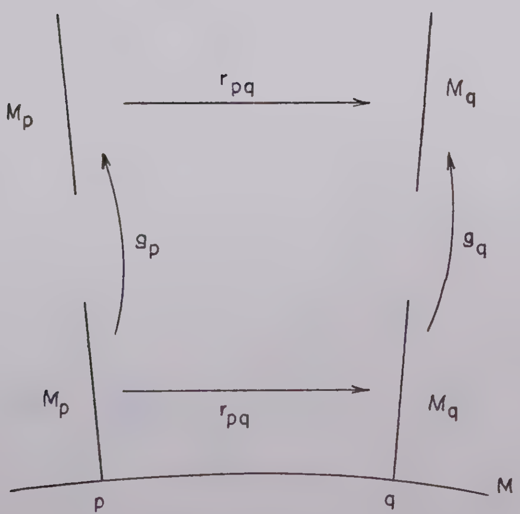


Fig. 1.2.3.

$h \in \mathcal{g}_p$. Clearly the stress tensor is intrinsic.

A tensor field μ is defined to be intrinsic if the element μ_p on $\mathcal{B}_p$ is intrinsic in the above sense and for any two points p,q , we have

$$\mu_q = \otimes r_{pq} \circ \mu_p \quad , \tag{1.2.10}$$

where $\otimes r_{pq}$ is a mapping on the tensor space, induced by the material isomorphism r_{pq} . Clearly, if μ_p is intrinsic, we can create an intrinsic field μ through the mappings r_{pq} . Since these mappings are assumed smooth, so is μ .

If (u_α, r_α) and (u_β, r_β) are two charts of $\mathcal{U}$ on a neighbourhood of p , then $r_\alpha(p) \circ r_\beta^{-1}(p)$ is a material isomorphism. Define $G_{\alpha\beta} : R^3 \to R^3$, $G_{\alpha\beta} \in \mathcal{G}$ by

$$G_{\alpha\beta} = r_\alpha \circ r_\beta^{-1} \quad . \tag{1.2.11}$$

The $G_{\alpha\beta}$ form the group of coordinate transformations. If $\frac{\partial}{\partial x^j}$ are the base vectors under r_β and $\frac{\partial}{\partial y^i}$ are the base vectors under r_α , then $G_{\alpha\beta} = ((\frac{\partial y^i}{\partial x^j}))$. We have

1. $G_{\alpha\alpha}|_p = I$,

2. $G_{\alpha\beta}|_p = G_{\beta\alpha}^{-1}|_p$,

3. $G_{\alpha\beta}|_p \circ G_{\beta\gamma}|_p = G_{\alpha\gamma}|_p$.

A point p is a solid point if there exists a local configuration $r_p \ni G_p \subset SO(3)$, otherwise p is a fluid point. In a materially uniform body $\mathcal{B}$, all points are solid if one is, so $\mathcal{B}$ is solid if $G \subset SO(3)$. $\mathcal{B}$ is isotropic if $SO(3) \subset G$. Hence for a solid isotropic body, $G = SO(3)$.

We make one further assumption, necessary for the development of the material geometry: the isotropy group is a closed Lie subgroup of $SL(\mathcal{B}_p)$. This assumption is satisfied by isotropic solid bodies in particular.

1.3 The Specialized Constitutive Equation.

We now show how the constitutive equation (1.1.17) can be reduced to a simple coordinate form. By the polar decomposition theorem, a matrix F , $\det F > 0$, may be factored uniquely:

$$F = RU = VR \quad , \tag{1.3.1}$$

where R is orthogonal and U,V are positive definite and symmetric. We define two tensors B and C , called the left and right Cauchy-Green tensors, respectively, by:

$$B = FF^T = VRR^TV^T = V^2 \quad , \tag{1.3.2a}$$

and

$$C = F^TF = U^TR^TRU = U^2 \quad . \tag{1.3.2b}$$

Now for a solid isotropic body, $\mathcal{G} = SO(3)$, so if we choose $G = R^T$ in (1.2.9), we get

$$\tau = \mathcal{R}(FR^T) = \mathcal{R}(VRR^T) = \mathcal{R}(V) = \mathcal{H}(B) \quad . \tag{1.3.3}$$

In preliminaries to their major work on isotropic functions, Rivlin [3] and Ericksen and Rivlin [1] used physical principles (cf. Appendix A) to show that the actual form of (1.3.3) for an isotropic elastic solid must be

$$\tau = f_0 I + f_1 B + f_2 B^2 \quad , \tag{1.3.4}$$

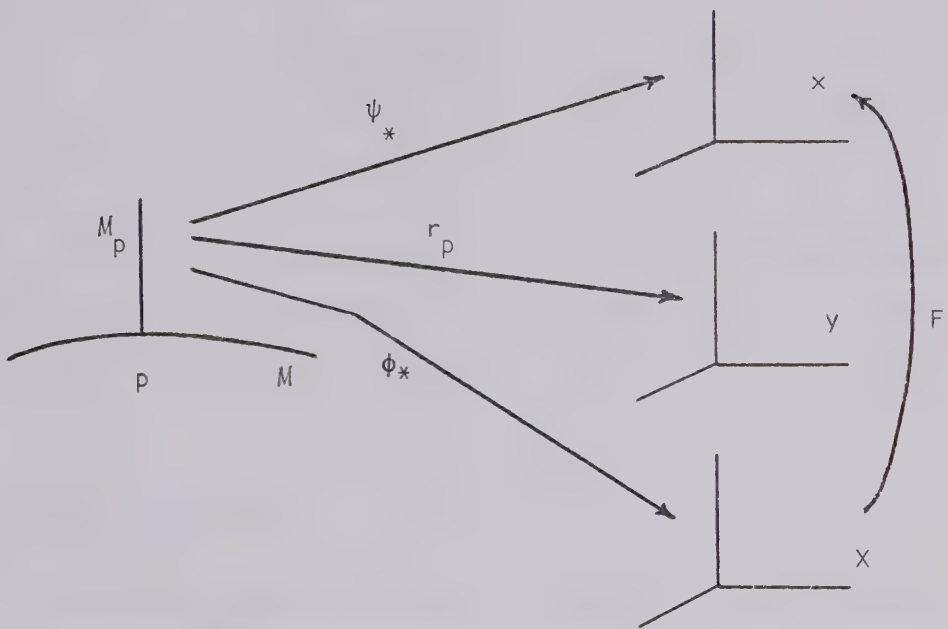
where the f's are functions of the three principal invariants of B.

We also state now, and discuss more fully in Section 1.6, that on a solid body we have an induced, intrinsic, Riemannian metric $\overline{g}$ which, relative to some local reference configuration r_p, has the form

$$\overline{g} = \delta^{ij} \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j} \quad . \tag{1.3.5}$$

Here the fact that r_p may possibly not be induced from a global configuration is important.

We assume that we have a global configuration ϕ which gives an X - system of coordinates. We denote the deformation configuration by ψ, giving an x - system of coordinates, and we also have base vectors $\frac{\partial}{\partial y^i}$ under r_p.



In the X - system

$$\bar{g} = \delta^{ij} \frac{\partial x^A}{\partial y^i} \frac{\partial x^B}{\partial y^j} \frac{\partial}{\partial X^A} \otimes \frac{\partial}{\partial X^B} = \bar{g}^{AB} \frac{\partial}{\partial X^A} \otimes \frac{\partial}{\partial X^B} \quad . \tag{1.3.6}$$

We also have

$$F = \psi_* \circ \phi_*^{-1} = \left(\left(\frac{\partial x^i}{\partial X^A}\right)\right) \quad , \tag{1.3.7}$$

and

$$F_\alpha = \psi_* \circ r_p^{-1} = \psi_* \circ \phi_*^{-1} \circ \phi_* \circ r_p^{-1} = F \circ \phi_* \circ r_p^{-1} = FK \quad . \tag{1.3.8}$$

$$\therefore \qquad B = F_\alpha F_\alpha^T = FKK^T F^T \quad , \tag{1.3.9}$$

or

$$B^{ij} = \frac{\partial x^i}{\partial X^A} \frac{\partial X^A}{\partial y^m} \frac{\partial X^B}{\partial y^n} \frac{\partial x^j}{\partial X^B} \delta^{mn}$$

$$= \frac{\partial x^i}{\partial X^A} \frac{\partial x^j}{\partial X^B} \bar{g}^{AB} \quad , \tag{1.3.10}$$

and no reference to r_p appears.

Now that we have an explicit representation for the stress tensor, we need the equilibrium equations. To derive these, we have to discuss covariant differentiation on a manifold.

1.4 Material Bundle Spaces.

Material bundle spaces are new differentiable manifolds over M which are determined by the response of the body B . We will describe their construction by reviewing the construction of the ordinary bundle spaces.

To discuss differentiation on M , we require two new manifolds: the tangent bundle, $T(M)$, and the bundle of linear frames, $G(M)$.

Briefly, a material tangent bundle is a sub-bundle of $T(M)$, characterized by the fact that its structure group is not $GL(3)$ but G , the isotropy group.

The restriction is often significant. Consider a spherical shell of a transversely isotropic material whose axis of transverse isotropy is in the radial direction at each point. Since the shell is merely a submanifold of R^3 , its geometric tangent bundle is trivial (if S denotes the shell, $T(S) = S \times R^3$). The material tangent bundle is equivalent to the tangent bundle of S^2 , which is not trivial (it is not a global product).

The geometric tangent bundle $T(M)$ over the base space M is a new differentiable manifold. It has the form

$$T(M) = \bigcup_{p \in M} M_p \quad . \tag{1.4.1}$$

Its genetic element is (p,v) with $p \in M$ and $v \in M_p$. There is a projection map

$$\bar{\pi} : T(M) \to M \ , \quad \bar{\pi}(p,v) = p \quad , \quad \bar{\pi}^{-1}(p) = M_p \quad . \tag{1.4.2}$$

We introduce charts (u_α, η_α) where u_α is an open set of M and η_α is a bundle map

$$\eta_\alpha : u_\alpha \times R^3 \to T(u_\alpha) \quad . \tag{1.4.3}$$

These maps are diffeomorphisms and can be described pointwise by:

$$\eta_{\alpha p} : \{p\} \times R^3 \to M_p \quad , \tag{1.4.4}$$

so that all fibres $\bar{\pi}^{-1}(p)$ are copies of R^3 .

If $p \in u_\alpha \cap u_\beta$, we define coordinate transformations

$$\eta_{\alpha p}^{-1} \circ \eta_{\beta p} : R^3 \to R^3 \quad , \tag{1.4.5}$$

and we require that these be smooth fields with values in $GL(3)$. We then call $GL(3)$ the <u>structure group</u>.

Consider charts $(u_\alpha, \psi_{\alpha *})$, $(u_\beta, \psi_{\beta *})$ on M , corresponding to coordinates $\{x^i\}$ and $\{y^j\}$, respectively, for $p \in M$. Then if $v \in M_p$,

and $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$, so that

$$\psi_{\alpha *}\big|_p (v) = (v^1, v^2, v^3) \quad , \tag{1.4.6}$$

we define

$$\eta_{\alpha p}(v^1, v^2, v^3) = v \quad . \tag{1.4.7}$$

Clearly then,

$$\eta_{\alpha p}^{-1} \circ \eta_{\beta p} = \left(\left(\left.\frac{\partial x^i}{\partial y^j}\right|_p\right)\right) \in GL(3) \quad , \tag{1.4.8}$$

and they form a smooth field on $u_\alpha \cap u_\beta$.

Corresponding to the atlas $V = \{(u_\alpha, \psi_{\alpha *}) : \alpha \in A\}$ for M , we construct a <u>pre-atlas</u> $\Theta' = \{(u_\alpha, \eta_\alpha) : \alpha \in A\}$, and hence a maximal atlas Θ for $T(M)$. In particular, since by (1.1.1) we require only one coordinate chart, we require only one bundle map for Θ' of $T(M)$. Note that $\psi_{\alpha *}$ is not a reference configuration unless the response functional is independent of the coordinates $\{x^i\}$, i.e. independent of the points in u_α .

A <u>material</u> <u>chart</u> is a chart $(u_\alpha, \eta_\alpha) \in \Theta$, $\ni$

$$r_\alpha(p,q) \equiv \eta_{\alpha p} \circ \eta_{\alpha q}^{-1} : M_q \to M_p \tag{1.4.9}$$

is a material isomorphism for all $p,q \in u_\alpha$. Two material charts (u_α, η_α) and (u_β, η_β) are compatible if

$$r_{\alpha\beta}(p,q) \equiv \eta_{\alpha p} \circ \eta_{\beta q}^{-1} : M_q \to M_p \tag{1.4.10}$$

is a material isomorphism for all $p \in u_\alpha$, $q \in u_\beta$. A material atlas is a maximal collection of pairwise-compatible material charts $\{(u_\alpha,\eta_\alpha) : \alpha \in A\}$, where the set $\{u_\alpha : \alpha \in A\}$ is an open cover of M .

Clearly not all charts in the geometric pre-atlas Θ' are material. If for any $p \in M$ we can find a material chart $(u_\alpha,\eta_\alpha) \in \Theta'$, where $p \in u_\alpha$, then B is called locally homogeneous. If there is a single such chart in Θ' for all $p \in M$, then B is globally homogeneous.

If we define

$$r_\alpha(p) = \eta_{\alpha p}^{-1} \quad , \tag{1.4.11}$$

then it is easy to show that material charts are in one-one correspondence with reference charts. Hence for all bodies that we consider here, a material atlas exists. If U denotes the reference atlas, we write $\eta(U)$ for the material atlas. If G is the isotropy group corresponding to U , then G is the structure group of the material tangent bundle.

A linear frame at $p \in M$ is an ordered basis

$$e(p) = \{e(p)_1, e(p)_2, e(p)_3 : e(p)_i \in M_p\} \quad . \tag{1.4.12}$$

$E(p)$ is the set of all linear frames at p , and

$$E(M) = \bigcup_{p \in M} E(p) \tag{1.4.13}$$

is called the bundle of linear frames, or the associated principal bundle of $T(M)$. $E(M)$ is a differentiable manifold, and as before, we have

$$\pi : E(M) \to M \ , \ \pi(p,e(p)) = p \ , \ \pi^{-1}(p) = E(p) \quad . \tag{1.4.14}$$

Clearly, if $e(p)$ is a linear frame, then

$$\hat{e}(p) = \{e(p)_i \ G^i_j \ : \ ((G^i_j)) = G \in GL(3)\} \tag{1.4.15}$$

is also a linear frame. We define

$$R_G : E(M) \to E(M) \quad \text{by} \quad R_G(e(p)) = \{e(p)_i \ G^i_j\} \quad . \tag{1.4.16}$$

Let $\hat{\imath}$ be the standard frame of R^3. Then

$$e_\alpha(p) = \eta_{\alpha p}(\hat{\imath}) = \{\eta_{\alpha p}(\underline{i}), \eta_{\alpha p}(\underline{j}), \eta_{\alpha p}(\underline{k})\}. \tag{1.4.17}$$

We define a chart (u_α, ξ_α) where the ξ_α are isomorphisms:

$$\xi_\alpha : u_\alpha \times GL(3) \to E(u_\alpha), \xi_\alpha(p,G) = R_G(e_\alpha(p)) \quad . \tag{1.4.18}$$

Then under a change of coordinates at $p \in u_\alpha \cap u_\beta$, we get

$$\xi_\alpha^{-1} \circ \xi_\beta(p,G) = (p, G_{\alpha\beta}|_p \circ G) \quad , \tag{1.4.19}$$

where, combining (1.4.11) and (1.2.11), $G_{\alpha\beta} = \eta_\alpha^{-1} \circ \eta_\beta$. Therefore,

$$\xi_\alpha^{-1} \circ \xi_\beta \equiv L_{G_{\alpha\beta}} \quad , \tag{1.4.20}$$

where L denotes left-translation. In particular, if we choose two charts (u_α, ξ_α) and (u_β, ξ_β) corresponding to $(u_\alpha, \psi_{\alpha *})$ and $(u_\beta, \psi_{\beta *}) \in V$, where the base vectors under $\psi_{\alpha *}$ are $\{\frac{\partial}{\partial x^i}\}$ and under $\psi_{\beta *}$ are $\{\frac{\partial}{\partial y^j}\}$,

then if

$$\xi_\alpha^{-1} \circ \xi_\beta(p,H) = (p,G) \tag{1.4.21}$$

and therefore

$$R_G(e_\alpha(p)) = R_H(e_\beta(p)) \quad , \tag{1.4.22}$$

we have

$$G^k_m \frac{\partial}{\partial x^k} = H^k_m \frac{\partial}{\partial y^k} = H^n_m \frac{\partial x^k}{\partial y^n} \frac{\partial}{\partial x^k} \quad , \tag{1.4.23}$$

so that

$$G^k_m = H^n_m \frac{\partial x^k}{\partial y^n} = \frac{\partial x^k}{\partial y^n} H^n_m \quad , \tag{1.4.24}$$

and the transformation is on the left. Note that for base vectors the transformation is on the right:

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} = \frac{\partial}{\partial y^k} \frac{\partial y^k}{\partial x^i} \quad . \tag{1.4.25}$$

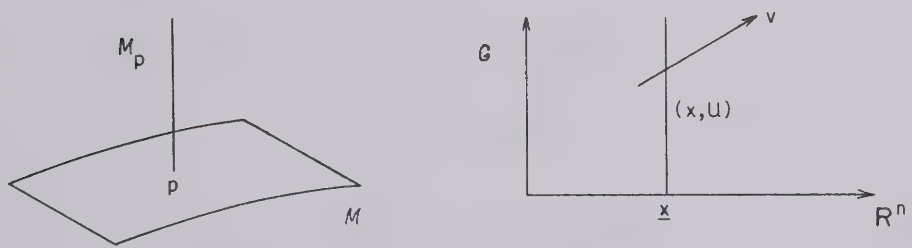
In equations (1.4.17) and (1.4.15) we may restrict ourselves to only those $\eta_{\alpha p}$ for which $(u_\alpha, \eta_\alpha) \in \eta(U)$, and to only those $G \in \mathcal{G}$, and thus arrive at a <u>bundle of reference frames</u>, denoted $E(U)$ or $E(M,U)$, which is a sub-bundle of $E(M)$. Locally, $E(u_\alpha, U)$ looks like $u_\alpha \times \mathcal{G}$.

1.5 Vector Fields, Connections, and Lie Derivatives.

In this section we give a fairly detailed discussion of vector fields on several manifolds, i.e. on M, on $\mathcal{G}$, and on $T(E(B))$, the

tangent bundle of the bundle of linear frames. Having defined vectors everywhere on G and $T(E(B))$ by means of left-invariant and fundamental fields, we talk about connections between fibres of $E(B)$, and use the concept of the Lie Derivative to introduce covariant differentiation of tensors on $T(M)$.

In dealing with the bundle of frames, it is helpful to keep the diagram below in mind.



The structure group G is used both to coordinatize frames as already described, and to act, as a group, on the vectors and frames. G is a Lie group, i.e. it is a differentiable manifold and there exist continuous maps

$$\Phi : G \times G \to G \ , \quad \Phi(a,b) = ab \tag{1.5.1a}$$

and

$$\Psi : G \to G \ , \quad \Psi(a) = a^{-1} \ . \tag{1.5.1b}$$

For generality, we take $G = GL(3)$ and assume that G is coordinatized, i.e. $a \in G$ is represented by $((A^i{}_j))$ in R^9.

The map

$$L_a : G \to G \ , \ a \in G \ , \ L_a(b) = ab \quad , \tag{1.5.2}$$

which was mentioned in (1.4.20), induces a map of vector spaces

$$L_{a*} : G_b \to G_{ab} \quad . \tag{1.5.3}$$

We say a vector field X is <u>left-invariant</u> if

$$L_{a*} X = X \quad . \tag{1.5.4}$$

Now if $X|_e$ is a tangent vector at the identity of G, we may construct a vector field X which is left-invariant, by defining the generic element $X|_a$ to be

$$X|_a = L_{a*} X|_e \quad . \tag{1.5.5}$$

Conversely, if X is left-invariant, then it must be of this form. The set of left-invariant vector fields has as many elements as the dimensions of G_e. In particular, the set of left-invariant vector fields on $GL(n)$ has n^2 elements.

There is a natural basis $\{\frac{\partial}{\partial x^i_j}\Big|_a\}$ of G_a. Define

$$V^p_q(a) = \frac{\partial}{\partial x^q_p}\Big|_a \tag{1.5.6}$$

and

$$E^p_q(a) = V^p_r(a)\ A^r_q \quad . \tag{1.5.7}$$

Clearly, (cf. (1.4.24)),

$$L_{a*}\ E^j_i\ (b) = E^j_i\ (ab) \quad . \tag{1.5.8}$$

On the other hand, cf. (1.4.25),

$$L_{a*}\ V^p_q\ (b) = V^p_r\ (ab)\ A^r_q \quad . \tag{1.5.9}$$

For later reference we note the effect of right-translation on E^j_i . If

$$R_a : G \to G\ ,\ R_a(b) = ba \quad , \tag{1.5.10}$$

then

$$R_{a*}(E^j_i(b)) = R_{a*}(V^j_p(b)B^p_i) = A^j_q\ V^q_p(ba)\ B^p_i \quad . \tag{1.5.11}$$

Now $V^q_p(ba) = \left.\dfrac{\partial}{\partial x^p_q}\right|_{ba}$, so

$$E^q_\ell(ba)\ A^{-1\,\ell}{}_m\ B^{-1}\ {}^m_p = V^q_p(ba) \quad , \tag{1.5.12}$$

and therefore

$$R_{a*}(E^j_i(b)) = A^j_q\ E^q_\ell\ (ba)A^{-1}\ {}^\ell_i \quad . \tag{1.5.13}$$

The set of left-invariant vector fields on G forms a Lie-Algebra, g, where the bracket operation is given by:

$$[Z,R]|_a = [Z^i_j(a)E^j_i(a), R^q_p(a)E^p_q(a)]$$

$$= (Z^i_j(a) \frac{\partial R^p_q}{\partial x^i_j}\Big|_a - R^i_j(a) \frac{\partial Z^p_q}{\partial x^i_j}\Big|_a) E^q_p(a) \quad , \tag{1.5.14}$$

so

$$L_{b*}[Z,R]|_a = (Z^i_j(ba) \frac{\partial R^p_q}{\partial x^i_j}\Big|_{ba} - R^i_j(ba) \frac{\partial Z^p_q}{\partial x^i_j}\Big|_{ba}) E^q_p(ba)$$

$$= [L_{b*}Z, L_{b*}R]|_{ba} \quad , \tag{1.5.15}$$

so the bracket operation is left-invariant.

We turn now to the bundle of frames. We choose a chart (u_α, ξ_α), corresponding to $(u_\alpha, \psi_{\alpha*})$, where the base-vectors under $\psi_{\alpha*}$ are $\{\frac{\partial}{\partial x^i}\}$. Then the identity of G corresponds to the base-vectors $\{\frac{\partial}{\partial x^i}\Big|_p\}$ under $\xi_{\alpha p}$. We have

$$\xi_{\alpha p}(e) = e(p) = \{\frac{\partial}{\partial x^i}\Big|_p\} \,. \tag{1.5.16}$$

Then

$$\xi_{\alpha p}(v) = \{\frac{\partial}{\partial x^i}\Big|_p \cdot v^i_j\} = \{\underline{v}_j\} \quad . \tag{1.5.17}$$

If $\hat{u} = (p,u)$ then $R_a \, \hat{u} = (p,ua) = \widehat{ua}$. (1.5.18)

If $z|_p \in M_p$, $z|_p = z^i \frac{\partial}{\partial x^i}\Big|_p$, then $z|_p$ can be raised from e to u on the fibre by

$$z\big|_p = z^i \, U^{-1}{}^k_i \, \underline{u}_k = z^k(u) \, \underline{u}_k \quad , \tag{1.5.19}$$

and therefore components of vectors transform according to:

$$z^i(ua) = A^{-1}{}^i_j \, z^j(u) \quad . \tag{1.5.20}$$

The extension to tensors is clear.

Now since $\xi_{\alpha p} : G \to E(p)$, we have an induced map

$$\xi_{\alpha p *} : X \in g \to \overline{x} \quad , \tag{1.5.21}$$

where $\overline{x}$ is a vector field on the fibre $E(p)$. Since coordinate changes under ξ correspond to left-translations, and X is left-invariant, $\overline{x}$ must be independent of charts, and the fields $\overline{x}$ form the <u>fundamental fields</u> on $E(M)$. We denote the set of these fields by $\overline{g}$. It can be shown that $\overline{g}$ is a Lie-Algebra with bracket

$$[\overline{x},\overline{y}] = \overline{[X,Y]} \tag{1.5.22}$$

The basis of $\overline{g}$ at $\hat{a}$ will be the set $\overline{E}^j_i(\hat{a})$. Denote

$$\overline{E}^j_i\,(\hat{e}) = \frac{\partial}{\partial e^i_j}\bigg|_{\hat{e}} \tag{1.5.23}$$

then

$$\overline{E}^i_j(\hat{u}) = U^p_j \, \frac{\partial}{\partial e^p_i}\bigg|_{\hat{u}} \quad . \tag{1.5.24}$$

Clearly, these fundamental fields lie in the fibre direction, and

$$\pi_*(\overline{E}^i_j) = 0 \quad . \tag{1.5.25}$$

We define the <u>vertical subspace</u> at $\hat{u} \in E(M)$ to be

$$V_{\hat{u}} = \bar{g}\big|_{\hat{u}} \quad . \tag{1.5.26}$$

Thus the vertical subspace at $\hat{u} = (p,u)$ is spanned by $\bar{E}^j_i(\hat{u})$ and has the dimension of $\bar{g}$. (If $G = GL(3)$, dim = 9 .)

We define a natural lift of the vector fields on M. Given an element z of the field with components $z^i[p]$ at p, we construct a curve $\sigma(t)$ on M through p which is the integral curve of

$$\sigma(t) = \{z^i[\sigma(t)]\} \quad , \quad \sigma(0) = p \quad . \tag{1.5.27}$$

If the coordinates of p are $\{x^i\}$, then to first order,

$$x^i \to x^i + z^i t \tag{1.5.28a}$$

$$\frac{\partial}{\partial x^i}\bigg|_p \to (\delta^k_i + z^k{}_{,i}t)\,\frac{\partial}{\partial x^k}\bigg|_{p+\Delta p} \tag{1.5.28b}$$

$$(x^i, U^k_s) \to (x^i + z^i t \;,\; U^r_x(\delta^k_r + z^k{}_{,r}t)) \tag{1.5.28c}$$

$$\therefore \quad z^i \frac{\partial}{\partial x^i}\bigg|_p \to z^i \frac{\partial}{\partial x^i}\bigg|_p + U^k_s\, z^r{}_{,k}\, \frac{\partial}{\partial e^r_s}\bigg|_{\hat{u}} \quad . \tag{1.5.29}$$

We define $\tilde{z}\big|_p$, the <u>natural lift</u> of $z\big|_p = z^i \frac{\partial}{\partial x^i}\big|_p$ to the frame $\hat{u}$, by

$$\tilde{z}\big|_p = z^i \frac{\partial}{\partial x^i}\bigg|_p + U^k_s\, z^r{}_{,k}\, \frac{\partial}{\partial e^r_s}\bigg|_{\hat{u}} \quad , \tag{1.5.30}$$

where ${}_{,k}$ denotes partial differentiation with respect to x^k at p .

We can now define, locally, the Lie Derivative of two vector fields z and y on M.

Suppose $y\big|_p = \mu^k \dfrac{\partial}{\partial x^k}\Big|_p = \mu^k U^{-1}{}^r{}_k \underline{u}_r = y^r \underline{u}_r$. (1.5.31)

Then we define

$$(L_z y)^r = \tilde{z}(y^r) = (z^i \frac{\partial}{\partial x^i}\Big|_p + U^k_s z^q{}_{,k} \frac{\partial}{\partial e^q_s}\Big|_{\hat{u}})(\mu^m U^{-1}{}^r{}_m)$$

$$= z^i \mu^m{}_{,i} + U^k_s z^q{}_{,k} \mu^m \frac{\partial U^{-1}{}^r{}_m}{\partial e^q_s}\Big|_{\hat{u}} \quad . \qquad (1.5.32)$$

Now

$$\frac{\partial U^{-1}{}^r{}_m}{\partial e^q_s}\Big|_{\hat{u}} = -U^{-1}{}^r{}_q U^{-1}{}^s{}_m \quad , \qquad (1.5.33)$$

so

$$(L_z y)^r = (z^i \mu^m{}_{,i} - \mu^i z^m{}_{,i}) U^{-1}{}^r{}_m \qquad (1.5.34a)$$

so

$$(L_z y)\big|_p = [z,y]\big|_p \quad . \qquad (1.5.34b)$$

The natural lift described here depends on the value of the vector field in a neighborhood of p. We now describe a method of lifting vectors which depends only on the value of $z\big|_p$.

A <u>connection</u> $\Gamma : E(\mathcal{B}) \to T(E(\mathcal{B}))$ is a map satisfying

$$\Gamma(\hat{u}) = H_{\hat{u}} \subset E_{\hat{u}}(\mathcal{B}) \quad , \qquad (1.5.35)$$

where

(i) $\pi_* H_{\hat{u}} = M_{\pi(\hat{u})}$

and

(ii) $R_{a*} H_{\hat{u}} = H_{\hat{u}a}$ for all $a \in G$.

$H_{\hat{u}}$ is a 3-dimensional subspace of $E_{\hat{u}}(\mathcal{B}) \equiv T_{\hat{u}}(E(\mathcal{B}))$, called the horizontal subspace. We have a decomposition

$$E_{\hat{u}}(\mathcal{B}) = V_{\hat{u}} \oplus H_{\hat{u}} \quad , \quad \text{where} \quad \pi_* V_{\hat{u}} = 0 \quad . \tag{1.5.36}$$

We need three vectors h_j spanning $H_{\hat{u}}$ such that

(i) $\pi_* h_j = h_j^k \underline{u}_k$,

and

(ii) $\{R_{a*} h_j\}$ span $H_{\hat{u}a}$. Define $B_r \in E_{\hat{u}}(\mathcal{B})$ by

$$\pi_* B_r = \underline{u}_r \quad . \tag{1.5.37}$$

Then $\{B_r, \overline{E}_j^i(u)\}$ span $E_{\hat{u}}(\mathcal{B})$.

Now define $\omega : E_{\hat{u}}(\mathcal{B}) \to V_{\hat{u}}$ as follows:

$$\text{If} \quad \overline{x} \in E_{\hat{u}}(\mathcal{B}) \ , \quad \text{set} \quad \omega(\overline{x}) = \omega_k^i(\overline{x}) \ \overline{E}_i^k(u) \quad . \tag{1.5.38}$$

The ω_k^i are the connection forms and we find

(i) $\omega(\overline{E}_i^k(u)) = \overline{E}_i^k(u)$,

(ii) $\omega(B_r) = 0$,

so we may write

$$\omega^i_k(\overline{E}^r_s(u)) = \delta^i_s\,\delta^r_k \quad , \tag{1.5.39a}$$

$$\omega^i(B_r) = 0 \quad . \tag{1.5.39b}$$

Next define $\theta : E_{\hat{u}}(\mathcal{B}) \to M_{\pi(\hat{u})}$ as follows:

$$\text{If} \quad \overline{x} \in E_{\hat{u}}(\mathcal{B}) \;, \quad \text{set} \quad \theta(\overline{x}) = \theta^i(\overline{x})\underline{u}_i \quad . \tag{1.5.40}$$

The three 1-forms θ^i are the <u>canonical 1-forms</u> and we find

(i) $\theta(\overline{E}^k_i(u)) = 0$

(ii) $\theta(B_r) = \underline{u}_r$,

so we may write

$$\theta^i(\overline{E}^r_s(u)) = 0 \quad , \tag{1.5.41a}$$

$$\theta^i(B_r) = \delta^i_r \quad . \tag{1.5.41b}$$

Now we look at the transformation properties. By property (ii) of the connection Γ ,

$$R_{a*}\, B_r\big|_{\hat{u}} = c^k_r\, B_k\big|_{\hat{u}a} \quad \text{for some} \quad c^k_r \tag{1.5.42}$$

and

$$\pi_*\, R_{a*}\, B_r\big|_{\hat{u}} = \pi_*\, B_r\big|_{\hat{u}} \quad , \tag{1.5.43}$$

because we stay in the same fibre. Then

$$\pi_* R_{a*} B_r\Big|_{\hat{u}} = \pi_* B_r\Big|_{\hat{u}} = \pi_* c_r^k B_k\Big|_{\hat{ua}} \quad . \tag{1.5.44}$$

From the definition of $\pi_* B_r\Big|_{\hat{u}}$ and $\pi_* B_k\Big|_{\hat{ua}}$ we deduce

$$c_r^k A_k^s = \delta_r^s \quad , \quad \therefore \quad c_r^k = A^{-1}{}_r^k \quad . \tag{1.5.45}$$

Therefore,

$$R_{a*} B_r\Big|_{\hat{u}} = A^{-1}{}_r^k B_k\Big|_{\hat{ua}} \quad . \tag{1.5.46}$$

As we saw previously

$$R_{a*} \bar{E}_i^j (u) = A_m^j \bar{E}_n^m (ua) A^{-1}{}_i^n \quad . \tag{1.5.47}$$

Now given a vector $\frac{\partial}{\partial x^i}\Big|_p$ we can lift it to the horizontal subspace at $\hat{e}$ by setting

$$\frac{\bar{\bar{\partial}}}{\partial x^i}\Big|_p = \frac{\partial}{\partial x^i}\Big|_p - \Gamma_{ki}^m \frac{\partial}{\partial e_k^m}\Big|_{\hat{e}} \quad , \tag{1.5.48}$$

or to the horizontal subspace at $\hat{u}$ by setting

$$\frac{\bar{\bar{\partial}}}{\partial x^i}\Big|_p = \frac{\partial}{\partial x^i}\Big|_p - \Gamma_{ki}^m U_r^k \frac{\partial}{\partial e_r^m}\Big|_{\hat{u}} \quad , \tag{1.5.49}$$

where the Γ_{ik}^m are functions determined by Γ . Clearly

$$B_r\Big|_{\hat{u}} = \lambda_r^s \frac{\bar{\bar{\partial}}}{\partial x^s}\Big|_p \quad , \tag{1.5.50}$$

and if $\pi_* B_r\big|_{\hat{u}} = \underline{u}_r = U^s_r \left.\frac{\partial}{\partial x^s}\right|_p$, from (1.5.37), then

$$B_r\big|_{\hat{u}} = U^s_r \left(\left.\frac{\partial}{\partial x^s}\right|_p - \Gamma^n_{ms} U^m_k \left.\frac{\partial}{\partial e^n_k}\right|_{\hat{u}}\right) = U^s_r \left.\frac{\overline{\overline{\partial}}}{\partial x^s}\right|_p \quad . \tag{1.5.51}$$

Using the duality property $\langle dx^i , \frac{\partial}{\partial x^j}\rangle = \delta^i_j$, we can easily show that

$$\theta^i = U^{-1}{}^i{}_s \, dx^s \tag{1.5.52}$$

and

$$\omega^p_k = U^{-1}{}^p{}_m (\Gamma^m_{ks} \, dx^s + de^m_k) \quad . \tag{1.5.53}$$

We now define the <u>covariant</u> <u>derivative</u>, $\nabla_z y$. If

$$z\big|_p = z^i \left.\frac{\partial}{\partial x^i}\right|_p , \; y\big|_p = \mu^m \left.\frac{\partial}{\partial x^m}\right|_p = \mu^m U^{-1}{}^r{}_m \underline{u}_r = y^r \underline{u}_r \; , \tag{1.5.54}$$

then

$$\begin{aligned} (\nabla_z y)^r &= \overline{\overline{z}}(y^r) \\ &= z^i \left(\frac{\partial}{\partial x^i} - \Gamma^m_{ki} U^k_q \frac{\partial}{\partial e^m_q}\right)(\mu^n U^{-1}{}^r{}_n) \\ &= z^i (\mu^m{}_{,i} + \Gamma^m_{ki} \mu^k) U^{-1}{}^r{}_m \quad , \end{aligned} \tag{1.5.55}$$

and since

$$\nabla_z y\big|_p = (\nabla_z y)^r \underline{u}_r \quad , \tag{1.5.56}$$

we define

$$\mu^m|_i = \mu^m_{,i} + \Gamma^m_{ki}\,\mu^k \qquad (1.5.57)$$

and write

$$\nabla_z y|_p = z^i\,\mu^m|_i\,\frac{\partial}{\partial x^m}\Big|_p \quad . \qquad (1.5.58)$$

Here (1.5.57) is the normal covariant derivative, and we may easily extend the concept to tensor quantities.

1.6 Material and Riemannian Connections.

The discussion above on connections may be easily specialized to define material connections. We restrict our bundle of frames $E(B)$ to $E(B,U)$, and our structure group $GL(3)$ to G , the isotropy group. Then G , being a closed Lie subgroup of $GL(3)$, has a Lie algebra $g(U)$, which is a Lie subalgebra of g . This induces a Lie subalgebra $\overline{g}(U)$ on $E(B,U)$, with corresponding fundamental fields which are restrictions of the fields considered above.

Suppose we have a curve $\lambda(t)$ through p on M . Let (u_α,ψ_α) be a coordinate chart around p and let the coordinates under ψ_α be $\{x^i\}$. If $\tilde{\lambda}(t)$ is a smooth natural lift of $\lambda(t)$, then the tangent vector to $\tilde{\lambda}(t)$ at $\hat{u} = (p,u)$ in $E(B,U)$ is, cf. (1.5.30):

$$\frac{\tilde{\partial}}{\partial t} = \dot{\lambda}^k\,\frac{\partial}{\partial x^k} + \dot{U}^r_s\,\frac{\partial}{\partial e^r_s} \quad . \qquad (1.6.1)$$

Now this vector is in $H_{\hat{u}}$ for some connection Γ iff

$$\dot{U}^r_s + \Gamma^r_{kn} U^k_s \dot{\lambda}^n = 0 \quad . \tag{1.6.2}$$

We then say the frame $\hat{u}$ is moved along $\lambda(t)$ by <u>parallel</u> <u>transport</u>. Clearly, any vector $v = v^k \frac{\partial}{\partial x^k} \in M_p$ is moved along $\lambda(t)$ by parallel transport iff, cf (1.5.58):

$$\dot{v}^k + \Gamma^k_{rs} v^r \dot{\lambda}^s = 0 \quad . \tag{1.6.3}$$

Parallel transport of vectors induces isomorphisms of the tangent spaces of M along $\lambda(t)$. If the induced isomorphisms are material isomorphisms, we say Γ is a <u>material</u> <u>connection</u>.

Now suppose we have a material atlas on M, and a smooth cross-section σ on (u_α, ψ_α), given by

$$\sigma(p) = \{F^i_j(p) \frac{\partial}{\partial x^i} : j = 1,2,3\} \quad . \tag{1.6.4}$$

This cross-section induces a map from M_p to the tangent space of $E(B,U)$ at $\sigma(p)$. We have

$$\sigma_* \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} + F^i_{k,j} \frac{\partial}{\partial e^i_k} \quad , \tag{1.6.5}$$

or

$$\sigma_* \frac{\partial}{\partial x^j} = \frac{\bar{\bar{\partial}}}{\partial x^j} + (F^m_{s,j} + \Gamma^m_{rj} F^r_s) F^{-1}{}^k_m \bar{\bar{E}}^s_k \quad . \tag{1.6.6}$$

Considering the tangent space at $\hat{e}$, corresponding to the identity of the

isotropy group, we see that Γ is a material connection iff

$$F^{-1}{}^{k}_{m}(F^{m}_{s,j} + \Gamma^{m}_{rj} F^{r}_{s}) \subset \overline{g}(u) \quad , \quad j = 1,2,3 \quad . \qquad (1.6.7)$$

Associated with a given connection are two further tensors, the curvature tensor

$$R^{i}_{jkm} = \Gamma^{i}_{jm,k} - \Gamma^{i}_{jk,m} + \Gamma^{i}_{sk}\,\Gamma^{s}_{jm} - \Gamma^{i}_{sm}\,\Gamma^{s}_{jk} \quad , \qquad (1.6.8)$$

and the torsion tensor

$$T^{i}_{jk} = \Gamma^{i}_{jk} - \Gamma^{i}_{kj} \quad . \qquad (1.6.9)$$

A given material connection is (locally) flat if for every $p \in \mathcal{M}$ there is a local coordinate chart in which the connection symbols vanish. It is known that a necessary and sufficient condition for a connection to be flat is that

$$R \equiv 0 \quad , \quad T \equiv 0 \quad . \qquad (1.6.10)$$

There is no reason why a given material connection should have either property.

Referring to the definition of local homogeneity, we see that if a body has a flat material connection, it is locally homogeneous, since there exists a coordinate chart around each point, in which the material and natural frames coincide. For solid bodies, we also have the converse, i.e., local homogeneity implies flatness.

Suppose we have a solid body with a reference atlas $\mathcal{U}$ relative to which the isotropy group is a subgroup of $SO(3)$. $\mathcal{U}$ is called an undistorted atlas. We may introduce a Riemannian metric on $\mathcal{U}$ by taking

$$G_{\mathcal{U}}\big|_p(u,v) = r_{\alpha,p}(u) \cdot r_{\alpha,p}(v) \tag{1.6.11}$$

where $u,v \in M_p$, $r_{\alpha,p} \equiv r_\alpha(p)$ is a reference chart, and the inner product is the ordinary Euclidean inner product on R^3. The orthogonality of the isotropy group shows that this is an intrinsic metric at p. Then we can extend $G_{\mathcal{U}}\big|_p$ to an intrinsic field on $\mathcal{U}$. Suppose we fix this atlas. Given any material connection, it follows that the covariant derivative of $G\big|_{\mathcal{U}}$ with respect to the material connection must vanish. We get

$$G_{ij}\big|_k = G_{ij,k} - \Gamma^m_{ik}\, G_{mj} - \Gamma^m_{jk}\, G_{im} = 0 \quad , \tag{1.6.12}$$

where the Γ's are material connection symbols. If the material connection is torsion-free, i.e., symmetric, then it is uniquely determined by G.

On the other hand, if we introduce an arbitrary intrinsic Riemannian metric $\overline{g}$, then we define the Riemannian connection to be the unique torsion-free connection such that

$$\overline{g}_{ij}\big|_k = \overline{g}_{ij,k} - \Gamma^m_{ik}\, \overline{g}_{mj} - \Gamma^m_{jk}\, \overline{g}_{im} = 0 \quad , \tag{1.6.13}$$

where in this case the Γ's are the ordinary Christoffel symbols. If this Riemannian connection is not material, then no material connection is torsion-free. If one of the material connections is torsion free, it must coincide with the Riemannian connection.

Looking at the example we gave earlier of a transversely isotropic spherical shell, we take a configuration such that the intrinsic Riemannian metric coincides with the Euclidean metric. The induced connection is not material, so there is no torsion-free material connection. Since any locally homogeneous body has a torsion-free material connection, this is an example of a body which is not locally homogeneous.

There may be several undistorted reference atlases for a given solid body, and so more than one intrinsic Riemannian metric can be induced in the above manner. Coleman and Noll have shown that if the solid is isotropic, the metric is unique to within a multiplicative constant. Since this constant will not affect the Christoffel symbols, a solid isotropic body has a unique Riemannian connection. Since this connection maps orthogonal frames to orthogonal frames, it is a material connection.

Generally the connection will not be flat. If the body is also locally homogeneous, then it is. Given any locally homogeneous solid body, isotropic or not, there is an undistorted atlas $\mathcal{U} = \{u_\alpha, r_\alpha\}$. By local homogeneity the field of reference maps r_α corresponds to the induced local configurations of the coordinate map ψ_α on u_α . The components of the intrinsic metric relative to ψ_α are δ_{ij} , so the connection is flat. As pointed out above it is also material.

1.7 Laminated Bodies.

An isotropic solid body $\mathcal{B}$ is called a laminated body if it is a disjoint union of a collection of two-dimensional submanifolds L_β , called the laminae,

$$\mathcal{B} = \bigcup_{\beta \in B} L_\beta \quad , \tag{1.7.1}$$

where B is an index set, and where each L_β has the property that in some neighbourhood of it, there exist configurations ψ_α whose induced local configurations carry the intrinsic metric G on L_β onto the Euclidean metric, i.e.,

$$G(u,v) = \psi_{\alpha *}(u) \cdot \psi_{\alpha *}(v) \tag{1.7.2}$$

for all $u,v \in M_p$, $p \in L_\beta \cap u_\alpha$, where u_α is the domain of ψ_α. A different set of ψ_α's may have to be chosen for each neighbouring lamina. The ψ_α's are called <u>initial configurations</u> for L_β.

Physically, we may think of the body as made up of a set of infinitely thin, "pre-stressed" sheets, or of a set of sheets deformed separately and then put together.

The body is inhomogeneous. There is in general no global configuration (deformation) of the body which will return it to an undeformed state.

It is shown in [7] that for certain classes of laminated bodies, there are universal solutions (possible deformations), which are the same as those for homogeneous bodies. One such class, and a universal solution for it, is given below.

A laminated cylinder is made up of thin cylindrical shells. We assume that there exists some set of initial configurations such that relative to a cylindrical coordinate system (R,θ,Z) the components of the

intrinsic metric G (or $\overline{g}$, to bring the notation into line with chapter 2), form the matrix

$$((\overline{g}^{rs})) = \begin{pmatrix} \overline{g}^{11} & 0 & 0 \\ 0 & \overline{g}^{22} & \overline{g}^{23} \\ 0 & \overline{g}^{32} & \overline{g}^{33} \end{pmatrix} \tag{1.7.3}$$

where the $\overline{g}$'s are functions only of R. We deal with incompressible bodies, so

$$\det\,((\overline{g}^{rs})) = 1/R^2 \quad . \tag{1.7.4}$$

From (1.3.4) we have a representation of the stress for an isotropic elastic solid. We may write it as

$$\tau = -pI + f_1 B + f_{-1} B^{-1} \tag{1.7.5}$$

where I is the metric tensor of the space and B is the deformation tensor relative to some fixed undistorted reference configuration. If we choose this configuration to be the one in which the intrinsic metric and the spatial metric coincide, then we have

$$\tau = -pI + f_1 \overline{g} + f_{-1} (\overline{g})^{-1} \tag{1.7.6}$$

We may write this simply as

$$\tau = -pI + S \quad , \tag{1.7.7}$$

where the tensor S has representation

$$((S^{ij})) = \begin{pmatrix} S^{11} & 0 & 0 \\ 0 & S^{22} & S^{23} \\ 0 & S^{32} & S^{33} \end{pmatrix} \tag{1.7.8}$$

and the S's are functions only of R.

Suppose we deform the body according to

$$r^2 = AR^2+B \quad , \quad \theta = C\Theta+DZ \quad , \quad z = E\Theta+FZ \quad , \tag{1.7.9}$$

where A,B,C,D,E,F are constants satisfying

$$A(CF-ED) = 1 \quad . \tag{1.7.10}$$

In the deformed coordinate system (r,θ,z) the intrinsic metric $\hat{g}$ has the components

$$\begin{aligned}
\hat{g}^{11} &= \left(\frac{AR}{r}\right)^2 \bar{g}^{11} \\
\hat{g}^{22} &= C^2\, \bar{g}^{22} + 2CD\, \bar{g}^{23} + D^2\, \bar{g}^{33} \\
\hat{g}^{23} &= \hat{g}^{32} = CE\, \bar{g}^{22} + (CF+DE)\bar{g}^{23} + DF\, \bar{g}^{33} \\
\hat{g}^{33} &= E^2\, \bar{g}^{22} + 2EF\, \bar{g}^{23} + F^2\, \bar{g}^{33} \\
&\hat{g}^{12} , \hat{g}^{13} , \hat{g}^{21} , \hat{g}^{31} = 0 \quad .
\end{aligned} \tag{1.7.11}$$

From (1.7.4) it is clear that

$$\hat{g} = \frac{A^2(CF-ED)^2}{r^2} = \frac{1}{r^2} \quad , \tag{1.7.12}$$

so we have incompressibility satisfied.

The equilibrium equations for a deformation are

$$\tau^{\alpha\beta}|_{\alpha} = 0 \quad . \tag{1.7.13}$$

Since we are dealing with an isotropic solid, the Riemannian connection is the material connection we need for covariant differentiation. Several of the τ's vanish by virtue of (1.7.7) and (1.7.8), and the S's are functions only of r, so (1.7.13) becomes

$$\frac{\partial}{\partial r}(-p) + \frac{dS^{11}}{dr} - r\left(-\frac{p}{r^2} + S^{22}\right) + \frac{1}{r}(-p + S^{11}) = 0 \tag{1.7.14a}$$

$$\frac{\partial}{\partial \theta}\left(-\frac{p}{r^2}\right) = 0 \tag{1.7.14b}$$

$$\frac{\partial}{\partial z}(-p) = 0 \quad . \tag{1.7.14c}$$

We see that this deformation is in equilibrium under a pressure given by

$$p = p(r) = S^{11} - \int^{r} \frac{\rho^2 S^{22} - S^{11}}{\rho} \, d\rho \quad . \tag{1.7.15}$$

CHAPTER 2

The Equations of Equilibrium for a Perturbation

2.1 Deformations.

We deal exclusively with simple, materially uniform, solid, isotropic bodies. Suppose we have an initial configuration X for a body $\mathcal{B}$, and a deformed configuration x where

$$x^r = x^r(X^A) \quad , \tag{2.1.1}$$

both coordinate systems being rectangular Cartesian. The deformation gradient F has representation

$$F = \left(\left(\frac{\partial x^r}{\partial X^A}\right)\right) \quad , \tag{2.1.2}$$

and the left Cauchy-Green tensor B has components

$$B^{rs} = \frac{\partial x^r}{\partial X^A}\frac{\partial x^s}{\partial X^B}\delta^{AB} \tag{2.1.3}$$

referred to the coordinates $\{x^i\}$.

In order to determine a standard way of examining small deformations on large, we bring this notation into line with that of Green and Zerna [2]. We must then consider general curvilinear coordinates. Let $\{\theta^\alpha\}$ denote such a set in the initial system with base vectors

$$g_\alpha = \frac{\partial X^A}{\partial \theta^\alpha}\frac{\partial}{\partial X^A} \quad , \tag{2.1.4}$$

and a metric

$$g_{\alpha\beta} = \frac{\partial X^A}{\partial \theta^\alpha} \frac{\partial X^B}{\partial \theta^\beta} \delta_{AB} \quad . \tag{2.1.5}$$

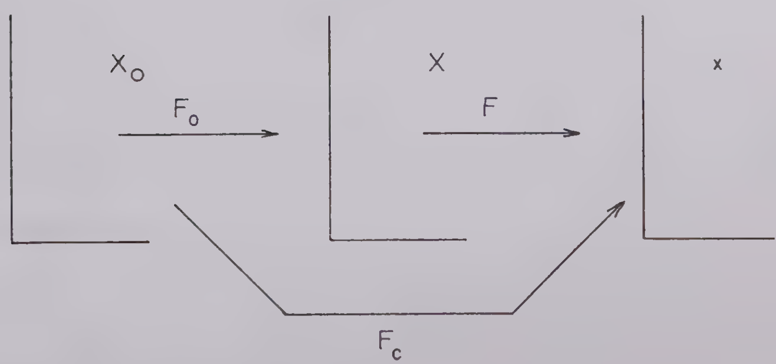
Under the deformation map from X to x the induced map of vectors gives a set of base vectors

$$G_\alpha = \frac{\partial x^r}{\partial \theta^\alpha} \frac{\partial}{\partial x^r} \tag{2.1.6}$$

and a corresponding metric

$$G_{\alpha\beta} = \frac{\partial x^r}{\partial \theta^\alpha} \frac{\partial x^s}{\partial \theta^\beta} \delta_{rs} \quad . \tag{2.1.7}$$

A state of deformation may already exist in the initial configuration, and some care is required in considering the composition of deformations. If we have three configurations X_o , X , and x and deformation gradients as shown diagrammatically below, then



$$F_c = F \cdot F_o \quad . \tag{2.1.8}$$

Let B_c denote the Cauchy-Green tensor in x due to the deformation from X_o to x, and B_o the tensor in X due to the deformation from X_o to X. Then

$$B_c^{rs} = \frac{\partial x^r}{\partial X^A} \frac{\partial x^r}{\partial X^B} B_o^{AB} \tag{2.1.9a}$$

$$(B_c^{-1})_{rs} = \frac{\partial X^A}{\partial x^r} \frac{\partial X^B}{\partial x^s} (B_o^{-1})_{AB} \tag{2.1.9b}$$

$$(B_c^2)^{rs} = \frac{\partial x^r}{\partial X^A} \frac{\partial x^m}{\partial X^B} \frac{\partial x^n}{\partial X^C} \frac{\partial x^s}{\partial X^D} \delta_{mn} B_o^{AB} B_o^{CD} \quad . \tag{2.1.9c}$$

To follow Green and Zerna, B_o will be given in terms of a coordinate system $\{\bar{\bar{\theta}}^i\}$ and B in terms of a system $\{\bar{\theta}^i\}$. In the case of a laminated solid the tensor B_o coincides with the induced metric $\bar{g}$ and we assume this is known in the initial state. To be definite we take the components of $\bar{g}$ to be $\bar{g}^{rs}$ relative to a given curvilinear coordinate system $\{\bar{\bar{\theta}}^i\}$ in the intial state. Then relative to the $\{\theta^\alpha\}$ in the deformed state, $\bar{g}$ has components $\hat{g}^{\alpha\beta}$ given by

$$\hat{g}^{\alpha\beta} = \frac{\partial\theta^\alpha}{\partial\bar{\bar{\theta}}^i} \frac{\partial\theta^\beta}{\partial\bar{\bar{\theta}}^j} \bar{g}^{ij} \quad . \tag{2.1.10}$$

Let B_c have components $\bar{B}^{\alpha\beta}$ relative to the convected coordinate system $\{\theta^\alpha\}$ in the deformed state. If we set $\{\theta^\alpha\} \equiv \{\bar{\theta}^i\}$ then some computation yields

$$\bar{B}^{\alpha\beta} = \hat{g}^{\alpha\beta} \tag{2.1.11a}$$

$$(\bar{B}^{-1})_{\alpha\beta} = (\hat{g}^{-1})_{\alpha\beta} \tag{2.1.11b}$$

$$(\overline{B}^2)^{\alpha\beta} = G_{\delta\rho}\, \hat{g}^{\alpha\delta}\, \hat{g}^{\rho\beta} \quad , \tag{2.1.11c}$$

and we find the invariants

$$I_1 = \operatorname{tr} B = \hat{g}^{\alpha\beta}\, G_{\alpha\beta} \tag{2.1.12a}$$

$$I_2 = \frac{1}{2}\{(\operatorname{tr} B)^2 - \operatorname{tr} B^2\} = G^{\alpha\beta}(\hat{g}^{-1})_{\alpha\beta}\, I_3 \tag{2.1.12b}$$

$$I_3 = \det B = G/\hat{g} \quad , \tag{2.1.12c}$$

where $G, \hat{g}$ are the determinants of $((G^{\alpha\beta}))$, $((\hat{g}^{\alpha\beta}))$ respectively.

2.2 Small Deformations on Large.

The deformed body is set in a curvilinear coordinate system $\{\theta^\alpha\}$. Suppose we impose an additional small deformation. We write the new position vector of a point p as

$$\overline{\underline{r}} = \underline{r} + \epsilon\, \underline{w}(\underline{r}) \quad , \tag{2.2.1}$$

where $\underline{r}$ is the former position vector, $\epsilon\, \underline{w}(\underline{r})$ is the deformation field, and ϵ is a small parameter. We derive the first order theory. In general, we denote a perturbed quantity Q by

$$\overline{Q} = Q + \epsilon\, \hat{\hat{Q}} \quad . \tag{2.2.2}$$

The new tangent vectors are

$$\overline{G}_\alpha = G_\alpha + \epsilon\, \hat{\hat{G}}_\alpha \quad . \tag{2.2.3}$$

Since

$$\frac{\partial \bar{r}}{\partial \theta^\alpha} = \frac{\partial r}{\partial \theta^\alpha} + \epsilon \frac{\partial w}{\partial \theta^\alpha} \quad , \tag{2.2.4}$$

we have

$$\bar{G}_\alpha = G_\alpha + \epsilon\, w^\rho\big|_\alpha\, G_\rho \quad , \tag{2.2.5}$$

and therefore

$$\hat{\hat{G}}_\alpha = w^\rho\big|_\alpha\, G_\rho \quad . \tag{2.2.6}$$

If we take the dot product of the base vectors, we find

$$\hat{\hat{G}}_{\alpha\beta} = w^\rho\big|_\alpha\, G_{\beta\rho} + w^\rho\big|_\beta\, G_{\alpha\rho} = w_{\beta|\alpha} + w_{\alpha|\beta} \quad . \tag{2.2.7}$$

Then we may show that

$$\hat{\hat{G}}^{\alpha\beta} = -G^{\alpha\beta}\, G^{\beta\sigma}\, \hat{\hat{G}}_{\rho\sigma} \quad , \tag{2.2.8}$$

$$\hat{\hat{G}}^\alpha = \hat{\hat{G}}^{\alpha\beta}\, G_\beta + G^{\alpha\beta}\, \hat{\hat{G}}_\beta = -w^\alpha\big|_\rho\, G^\rho \quad . \tag{2.2.9}$$

It may also be shown that

$$\hat{\hat{G}} = 2w^\rho\big|_\rho \quad G \quad , \tag{2.2.10}$$

and hence

$$\sqrt{\bar{G}} = \sqrt{G}\,(1 + \epsilon\, w^\rho\big|_\rho) \quad . \tag{2.2.11}$$

We may now determine the equilibrium equations. Let $\tau^{\alpha\beta}$ be the components of the stress tensor in the $\{\theta^{\alpha}\}$ system, and let

$$\tau^{\alpha} = \tau^{\alpha\beta} \sqrt{G}\, G_{\beta} \quad . \tag{2.2.12}$$

Then

$$\bar{\tau}^{\alpha} = \tau^{\alpha} + \epsilon\, \hat{\hat{\tau}}^{\alpha} = \bar{\tau}^{\alpha\beta}\sqrt{\bar{G}}\; \bar{G}_{\beta} \quad , \tag{2.2.13}$$

and after some computation

$$\hat{\hat{\tau}}^{\alpha} = \sqrt{G}\, (\hat{\hat{\tau}}^{\alpha\beta} + \tau^{\alpha\beta}\, w^{\rho}|_{\rho} + \tau^{\alpha\rho}\, w^{\beta}|_{\rho})\, G_{\beta} \quad . \tag{2.2.14}$$

We now introduce

$$\lambda^{\alpha\beta} = \hat{\hat{\tau}}^{\alpha\beta} + \tau^{\alpha\beta}\, w^{\rho}|_{\rho} + \tau^{\alpha\rho}\, w^{\beta}|_{\rho} \tag{2.2.15}$$

and write

$$\hat{\hat{\tau}}^{\alpha} = \lambda^{\alpha\beta} \sqrt{G}\; G_{\beta} \quad . \tag{2.2.16}$$

The λ's are not symmetrical in general.

The Cauchy condition for the perturbed state is given by

$$\int \bar{t}\, dS + \int_V \bar{\rho}(\bar{F}-\bar{f})dV = 0 \quad , \tag{2.2.17}$$

where $\bar{t}$, $\bar{F}$, $\bar{f}$ are the perturbed stress and body-force vectors, and $\bar{\rho}$ is the perturbed density. We may rewrite (2.2.17) as

$$\int_S \bar{\tau}^{\alpha\beta} \sqrt{\bar{G}}\, \bar{G}_\beta\, n_\alpha \, dS + \int_V \bar{\rho}(\bar{F}-\bar{f})dV = 0 \tag{2.2.18}$$

or

$$\int_S \sqrt{G}\, (\tau^{\alpha\beta} + \epsilon\, \lambda^{\alpha\beta})G_\beta\, n_\alpha \, dS + \int_V \bar{\rho}(\bar{F}-\bar{f})dV = 0 \quad , \tag{2.2.19}$$

from which we obtain

$$\int_V \frac{1}{\sqrt{G}} \frac{\partial}{\partial \theta^\alpha} [(\tau^{\alpha\beta} + \epsilon\, \lambda^{\alpha\beta})\, G_\beta \sqrt{G}]\, dV + \int_V \bar{\rho}(\bar{F}-\bar{f})dV = 0 \quad . \tag{2.2.20}$$

Since the body is an isotropic solid, the material and Riemannian connections coincide, so we may use the symmetry property of the connection, and the fact that

$$\frac{\partial}{\partial \theta^\alpha} \sqrt{G} = \sqrt{G}\, \Gamma^\rho_{\rho\alpha} \tag{2.2.21}$$

to deduce that

$$\int_V (\tau^{\alpha\beta}|_\alpha + \epsilon\, \lambda^{\alpha\beta}|_\alpha)G_\beta \, dV + \int_V \bar{\rho}(\bar{F}-\bar{f})dV = 0 \quad . \tag{2.2.22}$$

The perturbation was imposed on a stable deformation, so for that deformation

$$\int_V \tau^{\alpha\beta}|_\alpha\, G_\beta \, dV + \int_V \rho(F-f)dV = 0 \quad , \tag{2.2.23}$$

and therefore to first order

$$\int_V \lambda^{\alpha\beta}|_\alpha\, G_\beta \, dV + \int_V [\rho(\hat{\hat{F}}-\hat{\hat{f}}) + \hat{\hat{\rho}}(F-f)]dV = 0 \quad . \tag{2.2.24}$$

In particular, if the body is incompressible,

$$\hat{\hat{\rho}} = 0 \tag{2.2.25}$$

and the equilibrium equations become

$$\lambda^{\alpha\beta}|_{\alpha} + \rho(\hat{\hat{F}}^{\beta} - \hat{\hat{f}}^{\beta}) = 0 \quad . \tag{2.2.26}$$

The boundary conditions also have a simple form

$$\bar{t} = (\tau^{\alpha\beta} + \epsilon\, \lambda^{\alpha\beta})\, \sqrt{G}\, G_{\beta}\, n_{\alpha} = \bar{p} = (p^{\alpha} + \epsilon\, \hat{\hat{p}}^{\alpha})\, n_{\alpha} \quad , \tag{2.2.27}$$

and since there was previous equilibrium,

$$t = \tau^{\alpha\beta}\, \sqrt{G}\, G_{\beta}\, n_{\alpha} = p^{\alpha}\, n_{\alpha} \quad , \tag{2.2.28}$$

so

$$\lambda^{\alpha\beta}\, \sqrt{G}\, G_{\beta}\, n_{\alpha} = p^{\alpha}\, n_{\alpha} \quad , \tag{2.2.29}$$

or

$$\lambda^{\alpha\beta}\, \sqrt{G}\, G_{\beta} = \hat{\hat{p}}^{\alpha} \quad , \text{ for } \underline{n} = n_{\alpha} \quad . \tag{2.2.30}$$

In particular if $\hat{\hat{p}}^{\alpha} = 0$ for a fixed α, then

$$\lambda^{\alpha\beta} = 0 \quad , \quad \beta = 1,2,3 \quad . \tag{2.2.31}$$

We now digress, to study the form of the τ's and λ's in greater detail. For an isotropic solid we have, using (1.3.4) and (2.1.11),

$$\tau^{\alpha\beta} = -p\, G^{\alpha\beta} + f_1\, \hat{g}^{\alpha\beta} + f_2\, G_{\sigma\rho}\, \hat{g}^{\alpha\sigma}\, \hat{g}^{\rho\beta} \quad , \tag{2.2.32}$$

where the f's are functions of the principal invariants of $\hat{g}^{\alpha\beta}$. If the body is incompressible, then the f's are functions only of I_1 and I_2 .

In particular, (see Appendix I) we have a strain energy function W and

$$f_1 = \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial I_1}(I_1,I_2) + I_1 \frac{\partial W}{\partial I_2}(I_1,I_2)\right) \quad , \tag{2.2.33a}$$

$$f_2 = \frac{-2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}(I_1,I_2) \quad . \tag{2.2.33b}$$

If we define

$$\Phi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1} \quad , \tag{2.2.34a}$$

$$\Psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2} \quad , \tag{2.2.34b}$$

and

$$\tilde{B}^{\alpha\beta} = I_1 \hat{g}^{\alpha\beta} - G_{\rho\sigma} \hat{g}^{\alpha\rho} \hat{g}^{\sigma\beta} \quad , \tag{2.2.35}$$

then we may write τ in the form

$$\tau^{\alpha\beta} = -p\, G^{\alpha\beta} + \Phi\, \hat{g}^{\alpha\beta} + \Psi\, \tilde{B}^{\alpha\beta} \quad . \tag{2.2.36}$$

We may determine the perturbed form of this equation fairly easily. Considering the invariants in the form (2.1.12), and assuming incompressibility, we find

$$\hat{\hat{I}}_3 = \hat{\hat{G}} = 2\, w^\rho|_\rho\, G = 0 \tag{2.2.37a}$$

$$\hat{\hat{I}}_2 = (\hat{\hat{G}}^{\alpha\beta} I_3 + G^{\alpha\beta} \hat{\hat{I}}_3)\hat{g}^{-1}_{\alpha\beta} = \hat{\hat{G}}^{\alpha\beta} I_3\, \hat{g}^{-1}_{\alpha\beta} \tag{2.2.37b}$$

$$\hat{\hat{I}}_1 = \hat{g}^{\alpha\beta} \hat{\hat{G}}_{\alpha\beta} = \hat{g}^{\alpha\beta}(w_{\alpha|\beta} + w_{\beta|\alpha}) \quad , \tag{2.2.37c}$$

and therefore

$$\hat{\hat{\tilde{B}}}^{\alpha\beta} = \hat{g}^{\alpha\beta}\,\hat{\hat{I}}_1 - \hat{g}^{\alpha\sigma}\,\hat{g}^{\rho\beta}\,\hat{\hat{G}}_{\sigma\rho}$$

$$= (\hat{g}^{\alpha\beta}\,\hat{g}^{\rho\sigma} - \hat{g}^{\alpha\rho}\,\hat{g}^{\sigma\rho})(w_{\rho|\sigma} + w_{\sigma|\rho}) \quad . \tag{2.2.38}$$

Using a first order expansion of (2.2.34a) we find

$$\overline{\Phi} = \Phi + \epsilon\,\frac{\partial\Phi}{\partial I_\alpha}\,\hat{\hat{I}}_\alpha \quad , \tag{2.2.39}$$

and thus

$$\hat{\hat{\Phi}} = \frac{\partial\Phi}{\partial I_\alpha}\,\hat{\hat{I}}_\alpha \quad , \tag{2.2.40}$$

and in the same way

$$\hat{\hat{\Psi}} = \frac{\partial\Psi}{\partial I_\alpha}\,\hat{\hat{I}}_\alpha \quad . \tag{2.2.41}$$

From (2.2.36),

$$\overline{\tau}^{\alpha\beta} = \tau^{\alpha\beta} + \epsilon\,\hat{\hat{\tau}}^{\alpha\beta} = -(p+\epsilon\hat{\hat{p}})(G^{\alpha\beta} + \epsilon\,\hat{\hat{G}}^{\alpha\beta}) + (\Phi + \epsilon\hat{\hat{\Phi}})\,\hat{g}^{\alpha\beta}$$

$$+ (\Psi + \epsilon\,\hat{\hat{\Psi}})(\tilde{B}^{\alpha\beta} + \epsilon\,\hat{\hat{\tilde{B}}}^{\alpha\beta}) \quad , \tag{2.2.42}$$

and therefore the perturbation term is

$$\hat{\hat{\tau}}^{\alpha\beta} = -\hat{\hat{p}}\,G^{\alpha\beta} - p\,\hat{\hat{G}}^{\alpha\beta} + \hat{\hat{\Phi}}\,\hat{g}^{\alpha\beta} + \hat{\hat{\Psi}}\,\tilde{B}^{\alpha\beta} + \Psi(\hat{g}^{\alpha\beta}\hat{g}^{\rho\sigma} - \hat{g}^{\alpha\beta}\hat{g}^{\sigma\beta})(w_{\rho|\sigma} + w_{\sigma|\rho}) \quad . \tag{2.2.43}$$

The constitutive equation for a Mooney-Rivlin material is of the form (2.2.36), with Φ and Ψ constant. For a material of this type,

$$\hat{\hat{\Phi}} = \hat{\hat{\Psi}} = 0 \quad , \tag{2.2.44}$$

and (2.2.43) reduces to

$$\hat{\hat{\tau}}^{\alpha\beta} = -\hat{\hat{p}}\; G^{\alpha\beta} - p\; \hat{\hat{G}}^{\alpha\beta} + \Psi(\hat{g}^{\alpha\beta}\hat{g}^{\rho\sigma} - \hat{g}^{\alpha\rho}\; \hat{g}^{\sigma\beta})(w_{\rho|\sigma} + w_{\sigma|\rho}) \; . \tag{2.2.45}$$

Summarizing the main results which we will use for the particular material and deformation to be considered, we have

$$\hat{\hat{G}}_{\alpha\beta} = w_{\alpha|\beta} + w_{\beta|\alpha} \tag{2.2.7}$$

$$\hat{\hat{G}}^{\alpha\beta} = -G^{\alpha\rho}\; G^{\sigma\beta}\; \hat{\hat{G}}_{\rho\sigma} \tag{2.2.8}$$

$$\hat{\hat{I}}_3 = 0 = w^{\rho}|_{\rho} \tag{2.2.37}$$

$$\tau^{\alpha\beta} = -pG^{\alpha\beta} + \Phi\; \hat{g}^{\alpha\beta} + \Psi(\hat{g}^{\sigma\rho}\hat{g}^{\alpha\beta} - \hat{g}^{\alpha\rho}\hat{g}^{\sigma\beta})G_{\rho\sigma} \tag{2.2.36}$$

$$\hat{\hat{\tau}}^{\alpha\beta} = -\hat{\hat{p}}G^{\alpha\beta} - p\hat{\hat{G}}^{\alpha\beta} + \Psi(\hat{g}^{\alpha\beta}\hat{g}^{\rho\sigma} - \hat{g}^{\alpha\rho}\hat{g}^{\sigma\beta})\hat{\hat{G}}_{\rho\sigma} \tag{2.2.45}$$

$$\lambda^{\alpha\beta} = \hat{\hat{\tau}}^{\alpha\beta} + \tau^{\alpha\rho}\; w^{\beta}|_{\rho} \quad . \tag{2.2.15}$$

Equations of equilibrium:

$$\lambda^{\alpha\beta}|_{\alpha} + \rho(\hat{\hat{F}}^{\beta} - \hat{\hat{f}}^{\beta}) = 0 \quad . \tag{2.2.26}$$

Boundary conditions:

$$\sqrt{G}\; \lambda^{\alpha\beta}\; G_{\beta}n_{\alpha} = \hat{\hat{p}}^{\alpha}\; n_{\alpha} \quad . \tag{2.2.30}$$

2.3 A Particular Deformation.

We consider a laminated right-circular cylinder under vertical compression. If we choose cylindrical coordinates (r,θ,z) in the deformed configuration, and let $\{\theta^{\alpha}\}$ be (R,Θ,Z), we have a universal solution of the form (1.7.9), namely

$$r^2 = AR^2 \quad , \quad \theta = \Theta \quad , \quad z = \frac{1}{A} Z \quad . \tag{2.3.1}$$

The deformation metric $\hat{g}^{\alpha\beta}$ then takes the form (cf. (1.7.11))

$$((\hat{g}^{\alpha\beta})) = \begin{pmatrix} A\,\bar{g}^{11} & 0 & 0 \\ 0 & \bar{g}^{22} & \frac{1}{A}\,\bar{g}^{23} \\ 0 & \frac{1}{A}\,\bar{g}^{32} & \frac{1}{A^2}\,\bar{g}^{33} \end{pmatrix}$$

where the $\bar{g}^{\alpha\beta}$ are the components of the intrinsic metric in the intial state, and are functions only of R. Clearly the $\hat{g}^{\alpha\beta}$ are functions only of r. The spatial metric, of course, is

$$((G^{\alpha\beta})) = \operatorname{diag}\left(1 \,,\, \frac{1}{r^2} \,,\, 1\right) \quad . \tag{2.3.3}$$

As shown in section 1.7, the incompressibility condition is satisfied, i.e.

$$\det((\bar{g}^{\alpha\beta})) = \frac{1}{R^2} \quad , \quad \hat{g} = \frac{1}{r^2} \quad . \tag{2.3.4}$$

We now give the form of τ in terms of the $\bar{g}^{\alpha\beta}$.

We refer to (2.2.35) and obtain expressions for the components of $\tilde{B}^{\alpha\beta}$ which are not identically zero. After some algebra,

$$\tilde{B}^{11} = A\,\bar{g}^{11}(r^2\bar{g}^{22} + \frac{1}{A^2}\,\bar{g}^{33})$$

$$\tilde{B}^{22} = A\,\bar{g}^{11}\,\bar{g}^{22} + \frac{1}{Ar^2\,\bar{g}^{11}} \tag{2.3.5}$$

$$\tilde{B}^{23} = \tilde{B}^{32} = \bar{g}^{11}\,\bar{g}^{23}$$

$$\tilde{B}^{33} = \frac{1}{A}\,(\bar{g}^{11}\,\bar{g}^{33} + \frac{1}{\bar{g}^{11}}) \quad .$$

When these results are substituted into (2.2.36), the non-vanishing $\tau^{\alpha\beta}$ become

$$\tau^{11} = -p + \Phi\,A\,\bar{g}^{11} + \Psi\,A\,\bar{g}^{11}(r^2\,\bar{g}^{22} + \frac{1}{A^2}\,\bar{g}^{33})$$

$$\tau^{22} = -\frac{p}{r^2} + \Phi\,\bar{g}^{22} + \Psi(A\,\bar{g}^{11}\,\bar{g}^{22} + \frac{1}{Ar^2\,\bar{g}^{11}}) \tag{2.3.6}$$

$$\tau^{23} = \tau^{32} = \frac{\Phi}{A}\,\bar{g}^{23} + \Psi\,\bar{g}^{11}\,\bar{g}^{23}$$

$$\tau^{33} = -p + \frac{\Phi}{A^2}\,\bar{g}^{33} + \frac{\Psi}{A}\,(\bar{g}^{11}\,\bar{g}^{33} + \frac{1}{\bar{g}^{11}}) \quad .$$

As shown in section 1.7, this deformation is in equilibrium under a pressure given by

$$p = S^{11} + \int^r \frac{\rho^2\,S^{22} - S^{11}}{\rho}\,d\rho \quad , \tag{1.7.15}$$

where for this particular material and deformation

$$S^{11} = \Phi\,A\,\bar{g}^{11} + \Psi\,A\,\bar{g}^{11}\,(r^2\,\bar{g}^{22} + \frac{1}{A^2}\,\bar{g}^{33}) \tag{2.3.7a}$$

$$S^{22} = \Phi\,\bar{g}^{22} + \Psi(A\,\bar{g}^{11}\,\bar{g}^{22} + \frac{1}{Ar^2\,\bar{g}^{11}}) \quad . \tag{2.3.7b}$$

We write the integrand as

$$Q = \frac{r^2 S^{22} - S^{11}}{r} \quad , \tag{2.3.8}$$

and therefore from (2.3.7),

$$rQ = \Phi(r^2 \bar{g}^{22} - A\,\bar{g}^{11}) + \frac{\Psi}{A}\left(\frac{1}{\bar{g}^{11}} - \bar{g}^{11}\,\bar{g}^{33}\right) \quad . \tag{2.3.9}$$

For a compression in equilibrium, then, the stress is given by

$$\begin{aligned} \tau^{11} &= -\int^{r} Q\,dr \\ r^2\tau^{22} &= \tau^{11} + rQ \\ r\tau^{23} &= r\tau^{32} = r\,\frac{\Phi}{A}\,\bar{g}^{23} + r\,\Psi\,\bar{g}^{11}\,\bar{g}^{23} \\ \tau^{33} &= \tau^{11} + \Phi\left(\frac{1}{A^2}\,\bar{g}^{33} - A\,\bar{g}^{11}\right) + \Psi\left(\frac{1}{A\bar{g}^{11}} - Ar^2\,\bar{g}^{11}\bar{g}^{22}\right) \quad . \end{aligned} \tag{2.3.10}$$

We wish to impose a perturbation on this compression, but the general system (2.3.10) gives perturbation equations of unreasonable complexity. We make some simplifying assumptions first. We assume

$$\Psi \equiv 0 \quad . \tag{2.3.11}$$

We also diagonalize the intrinsic metric by setting

$$\bar{g}^{23} \equiv \bar{g}^{32} \equiv 0 \quad . \tag{2.3.12}$$

Further, we assume the form of the intrinsic metric to be

$$((\bar{g}^{\alpha\beta})) = \mathrm{diag}(\bar{g}^{11}, \frac{\bar{g}^{11}}{R^2}, \frac{1}{(\bar{g}^{11})^2}) \quad . \tag{2.3.13}$$

With this choice of the metric, we see that the τ's are diagonalized. If we put (2.3.13) into (2.3.9), we get

$$rQ = 0 \quad , \tag{2.3.14}$$

whatever the values of Φ and Ψ, and hence τ^{33} is the only non-vanishing component

$$\tau^{33} = \Phi(\frac{1}{(A\bar{g}^{11})^2} - A\,\bar{g}^{11}) \quad . \tag{2.3.15}$$

We let

$$f = A\,\bar{g}^{11} \quad , \tag{2.3.16}$$

and substitute in (2.3.15), (2.3.7) and (1.7.15) to obtain

$$\tau^{33} = \Phi(\frac{1}{f^2} - f) \tag{2.3.17}$$

$$p = \Phi\, f \quad . \tag{2.3.18}$$

In the case of a homogeneous body,

$$\bar{g}^{11} = 1 \quad , \quad f = A \quad . \tag{2.3.19}$$

Suppose now that a small perturbing force due to gravity acts on the cylinder. It has components

$$(\hat{f}^{\beta}) = (0,0,-g) \quad . \tag{2.3.20}$$

We assume that the perturbation has vertical and radial effects, but no angular effect, so that the deformation field is given by the vector

$$(w^{\beta}) = (u,0,v) \quad , \tag{2.3.21}$$

where u and v are functions of r and z. We can now apply the perturbation theory of the last section to obtain the new equilibrium equations.

From (2.2.7),

$$\begin{aligned}
\hat{\hat{G}}_{11} &= 2\,\frac{\partial u}{\partial r} \\
\hat{\hat{G}}_{13} &= \hat{\hat{G}}_{31} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \\
\hat{\hat{G}}_{22} &= 2\,ur \\
\hat{\hat{G}}_{33} &= 2\,\frac{\partial v}{\partial z} \quad ,
\end{aligned} \tag{2.3.22}$$

and from (2.2.8)

$$\begin{aligned}
\hat{\hat{G}}^{11} &= -2\,\frac{\partial u}{\partial z} \\
\hat{\hat{G}}^{13} &= \hat{\hat{G}}^{31} = -\,\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} \\
\hat{\hat{G}}^{22} &= -\,\frac{2u}{r^3} \\
\hat{\hat{G}}^{33} &= -2\,\frac{\partial v}{\partial z} \quad .
\end{aligned} \tag{2.3.23}$$

Using the expressions for Ψ, p and τ^{33} which we have obtained previously, we reduce the formulas for the $\hat{\hat{\tau}}^{\alpha\beta}$ and $\lambda^{\alpha\beta}$ to:

$$\hat{\hat{\tau}}^{\alpha\beta} = -\hat{\hat{p}}\, G^{\alpha\beta} - \Phi\, f\, \hat{\hat{G}}^{\alpha\beta} \tag{2.3.24}$$

$$\lambda^{\alpha\beta} = \hat{\hat{\tau}}^{\alpha\beta} \quad , \quad \alpha = 1,2 \tag{2.3.25a}$$

$$\lambda^{3\beta} = \hat{\hat{\tau}}^{3\beta} + \Phi(\frac{1}{f^2} - f) w^{\beta}|_z \quad . \tag{2.3.25b}$$

The explicit expressions are:

$$\lambda^{11} = \hat{\hat{\tau}}^{11} = -\hat{\hat{p}} + 2\Phi f \frac{\partial u}{\partial r}$$

$$\lambda^{13} = \hat{\hat{\tau}}^{13} = \Phi f \ (\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r})$$

$$\lambda^{22} = \hat{\hat{\tau}}^{22} = -\frac{\hat{\hat{p}}}{r^2} \div 2\Phi f \frac{u}{r^2} \tag{2.3.26}$$

$$\lambda^{31} = \Phi(f \frac{\partial v}{\partial r} + \frac{1}{f^2}\frac{\partial u}{\partial z})$$

$$\lambda^{33} = -\hat{\hat{p}} + \Phi(f + \frac{1}{f^2}) \frac{\partial v}{\partial z} \quad .$$

The equilibrium equation

$$\lambda^{\alpha 2}|_{\alpha} = 0 \tag{2.2.26b}$$

is satisfied identically, while the remaining two take the form

$$\frac{\partial}{\partial r}(-\hat{\hat{p}} + 2\Phi f \frac{\partial u}{\partial r}) + \frac{1}{r}(-\hat{\hat{p}} + 2\Phi f \frac{\partial u}{\partial r}) - r\,(\frac{-\hat{\hat{p}}}{r^2} + 2\Phi f \frac{u}{r^3})$$
$$+ \frac{\partial}{\partial z}(\Phi(f \frac{\partial v}{\partial r} + \frac{1}{f^2}\frac{\partial u}{\partial z})) = 0 \quad . \tag{2.3.27}$$

$$\frac{\partial}{\partial r}(\Phi f(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r})) + \frac{1}{r}(\Phi f(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r})) + \frac{\partial}{\partial z}(-\hat{\hat{p}} + \Phi(f + \frac{1}{f^2})\frac{\partial v}{\partial z}) = -\rho g \quad . \tag{2.3.28}$$

We also have the incompressibility condition (2.2.37):

$$w^{\rho}\big|_{\rho} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0 \quad . \tag{2.3.29}$$

If we take the partial derivatives of (2.3.29) with respect to r and z, the resulting identities may be substituted into (2.3.27) and (2.3.28). The set of equations then becomes:

$$\frac{\partial \hat{\hat{p}}}{\partial r} = \Phi \frac{df}{dr}\frac{\partial u}{\partial r} + \frac{\Phi}{r}\frac{\partial}{\partial r}\left(rf \frac{\partial u}{\partial r}\right) - \Phi f \frac{u}{r^2} + \frac{\Phi}{f^2}\frac{\partial^2 u}{\partial z^2} \quad , \tag{2.3.30a}$$

$$\frac{\partial \hat{\hat{p}}}{\partial z} = \Phi \frac{df}{dr}\frac{\partial u}{\partial z} + \frac{\Phi}{r}\frac{\partial}{\partial r}\left(rf \frac{\partial v}{\partial r}\right) + \frac{\Phi}{f^2}\frac{\partial^2 v}{\partial z^2} - \rho g \tag{2.3.30b}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0 \quad . \tag{2.3.30c}$$

These equations, together with appropriate boundary conditions, determine the perturbations u, v uniquely in terms of f, although not in a closed form.

CHAPTER 3

Compression of a Laminated Cylinder Under Gravity

3.1 Description.

Consider a vertical right-circular cylinder of initial height h_1, radius r_1, supported at its horizontal faces by planes. We assume that no friction exists at these faces. The cylinder is laminated in such a way that the intrinsic metric coincides with the one given in (2.3.13).

We first suppose the cylinder to be compressed to height h_0 with the ends maintained in a horizontal position. Essentially this problem was solved in the previous chapter. The stress system is given by (2.3.10). Our purpose here is to investigate the effect of a small perturbation on this solution.

We refer to a paper of Vaughan [5] in which a similar investigation was carried out for a homogeneous cylinder under gravity. The basic assumption is that gravity effects are small and can be treated by first-order perturbation theory. We may compare our results with those of Vaughan [5] by setting our intrinsic metric equal to the ordinary spatial metric. As we shall see, our solutions reduce to those in [5] if we deal with the homogeneous case.

We review some of the results derived in chapter 2. The compression is given by (cf. (2.3.13)):

$$r^2 = \frac{h_1}{h_0} R^2 \quad , \quad \theta = \Theta \quad , \quad z = \frac{h_0}{h_1} Z \quad . \tag{3.1.1}$$

The intrinsic metric is

$$((\bar{g}^{\alpha\beta})) = \text{diag}\ (\bar{g}^{11}\ ,\ \frac{\bar{g}^{11}}{R^2}\ ,\ \frac{1}{(\bar{g}^{11})^2}) \tag{2.3.13}$$

and we define f, a positive function of r, by

$$f = \frac{h_1}{h_0}\,\bar{g}^{11} \quad . \tag{3.1.2}$$

If g is the constant perturbing gravity force, and u,v are the horizontal and vertical perturbations, respectively, then the perturbation is in equilibrium under a pressure field $\hat{\hat{p}}$ given by

$$\frac{\partial\hat{\hat{p}}}{\partial r} = \Phi\,\frac{df}{dr}\,\frac{\partial u}{\partial r} + \frac{\Phi}{r}\,\frac{\partial}{\partial r}\,(rf\,\frac{\partial u}{\partial r}) - \Phi\,f\,\frac{u}{r^2} + \frac{\Phi}{f^2}\,\frac{\partial^2 u}{\partial z^2} \tag{2.3.30a}$$

$$\frac{\partial\hat{\hat{p}}}{\partial z} = \Phi\,\frac{df}{dr}\,\frac{\partial u}{\partial z} + \frac{\Phi}{r}\,\frac{\partial}{\partial r}\,(rf\,\frac{\partial v}{\partial r}) + \frac{\Phi}{f^2}\,\frac{\partial^2 v}{\partial z^2} - \rho g \quad . \tag{2.3.30b}$$

The incompressibility condition is given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0 \quad . \tag{2.3.30c}$$

The boundary conditions are given by (2.2.30). There are no forces acting on the curved face $r = r_o$, so using (2.3.26) we have

$$\lambda^{11} = -\hat{\hat{p}} + 2\Phi f\,\frac{\partial u}{\partial r} = 0 \quad , \quad r = r_o \tag{3.1.3a}$$

$$\lambda^{13} = \Phi f\,(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}) = 0\ , \quad r = r_o \quad . \tag{3.1.3b}$$

We must also have

$$u(0,z) \equiv 0 \quad . \tag{3.1.4}$$

The ends of the cylinder are frictionless and are maintained horizontal by the action of the planes, so we have

$$\frac{\partial v}{\partial r}(r,h_0) = 0 \tag{3.1.5a}$$

$$v(r,0) = 0 \quad . \tag{3.1.5b}$$

The problem then is to solve (2.3.30) for $\hat{\hat{p}}, u$, and v , subject to the boundary conditions (3.1.3), (3.1.4), and (3.1.5).

3.2 Separation of Variables and the Compatibility Condition.

We satisfy (2.3.30c) by setting

$$u = \frac{T(r)}{r} Z(z) \quad , \quad v \doteq P(r)\, X(z) \quad . \tag{3.2.1}$$

Substituting in (2.3.30c), we find

$$\frac{T'(r)}{r} Z(z) + P(r)\, X'(z) = 0 \quad . \tag{3.2.2}$$

Therefore,

$$X' \equiv Z \quad , \quad P \equiv -\frac{T'}{r} \tag{3.2.3}$$

$$u = \frac{T}{r} Z \quad , \quad v = -\frac{T'}{r} \int^z Z \quad , \tag{3.2.4}$$

where the integration is carried out with respect to z . The equilibrium equations (2.3.30a,b) now become

$$\frac{1}{\Phi}\frac{\partial\hat{\hat{p}}}{\partial r} = [2f'(\frac{T}{r})' + f(\frac{T'}{r})']Z + \frac{T}{rf^2}Z'' \qquad (3.2.5a)$$

$$\frac{1}{\Phi}\frac{\partial\hat{\hat{p}}}{\partial z} = [f'(\frac{T}{r}) - \frac{T'}{rf^2}]Z' - \frac{1}{r}[rf(\frac{T'}{r})']'\int^z Z - \frac{\rho g}{\Phi}\,. \qquad (3.2.5b)$$

We now derive a compatibility condition which must be satisfied if the equations (3.2.5) are to be self-consistent. Integrating (3.2.5b) with respect to z gives

$$\frac{1}{\Phi}\hat{\hat{p}} = [f'\frac{T}{r} - \frac{T'}{rf^2}]Z - \frac{1}{r}[rf(\frac{T'}{r})']'\int^z\int^z Z - \frac{\rho g}{\Phi}z + \eta(r)\ , \qquad (3.2.6)$$

and differentiating this with respect to r ,

$$\frac{1}{\Phi}\frac{\partial\hat{\hat{p}}}{\partial r} = [f'\frac{T}{r} - \frac{T'}{rf^2}]'Z - [\frac{1}{r}(rf\{\frac{T'}{r}\}')']'\int^z\int^z Z + \eta' \quad . \qquad (3.2.7)$$

The compatibility condition is obtained from (3.2.7) and (3.2.5a). After some reduction, we find that we must have

$$[(\{f + \frac{1}{f^2}\}\frac{T'}{r})' - (rf')'\frac{T}{r^2}]Z + \frac{TZ''}{rf^2} + [\frac{1}{r}(rf\{\frac{T'}{r}\}')']'\int^z\int^z Z - \eta'(r) = 0 \quad .$$

(3.2.8)

We now study the possible forms that the function Z may assume, and reduce this equation to an ordinary differential equation for T .

3.3 Polynomial Solutions.

If Z is a polynomial of degree ν , then $\int^z\int^z Z$ is of degree $\nu+2$. To satisfy (3.2.8), the two highest powers of z must vanish, therefore

$$[\frac{1}{r}(rf\{\frac{T'}{r}\}')']' = 0 \quad . \tag{3.3.1}$$

The only solutions of (3.3.1) which satisfy (3.1.4) are

$$T = \alpha r^2 + \beta\int_0^r \rho\int_0^\beta \frac{\sigma}{f}\,d\sigma\,d\rho \quad , \tag{3.3.2}$$

where α and β are arbitrary constants.

It is impossible to choose α,β so as to make $[(\{f + \frac{1}{f^2}\}\frac{T'}{r})' - (rf')'\frac{T}{r^2}] \equiv 0$ for arbitrary f, so that if Z explicity contains a power of z, the identity (3.2.8) cannot be satisfied. However, if Z is a constant function, (3.2.8) becomes

$$[(\{f + \frac{1}{f^2}\}\frac{T'}{r})' - (rf')'\frac{T}{r^2}] - \eta'(r) = 0 \quad , \tag{3.3.3}$$

using (3.3.2), and since η is arbitrary, this condition can be met.

Therefore we have as possible deformations

$$u = \alpha r + \frac{\beta}{r}\int_0^r \rho\int_0^\rho \frac{\sigma}{f}\,d\sigma\,d\rho \quad , \tag{3.3.4a}$$

$$v = -(2\alpha + \beta\int_0^r \frac{\rho}{f}\,d\rho)(z+\gamma) \quad , \tag{3.3.4b}$$

where α,β are arbitrary constants. In the homogeneous case where f is a constant, these deformations reduce to

$$u = \alpha r + \beta r^3 \quad , \tag{3.3.5a}$$

$$v = -(2\alpha + 4\beta r^2)(z+\gamma) \quad , \tag{3.3.5b}$$

which agree with those of Vaughan [5]. (To satisfy (3.1.5), set $\beta = \gamma = 0$).

A more general solution of the compatibility condition can be obtained by considering sums of polynomials. Set

$$u = \sum_1^n \frac{T_m}{r} Z_m \quad , \quad v = - \sum_1^n \frac{T'_m}{r} \int^z Z_m \quad , \tag{3.3.6}$$

where the Z's are polynomials satisfying

$$\deg Z_m = 2n-2m \quad \text{or} \quad \deg Z_m = 2n-2m+1 \quad . \tag{3.3.7}$$

If we define operators A , B , C by

$$A \equiv \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rf\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\right\}\right)\right] \quad , \tag{3.3.8a}$$

$$B \equiv \frac{d}{dr}\left[\left(f + \frac{1}{f^2}\right)\frac{1}{r}\frac{d}{dr}\right] - \frac{[rf']'}{r^2} \quad , \tag{3.3.8b}$$

$$C \equiv \frac{1}{rf^2} \quad , \tag{3.3.8c}$$

then the compatibility condition takes the form (cf (3.2.8)):

$$\begin{aligned} A[T_1]\int^z\int^z Z_1 &+ \left(B[T_1]Z_1 + A[T_2]\int^z\int^z Z_2\right) \\ &+ \sum_1^{n-2}\left(C[T_m]Z''_m + B[T_{m+1}]Z_{m+1} + A[T_{m+2}]\int^z\int^z Z_{m+2}\right) \\ &+ \left(C[T_{n-1}]Z''_{n-1} + B[T_n]Z_n\right) + C[T_n]Z''_n - \eta'(r) = 0 \quad . \end{aligned} \tag{3.3.9}$$

From (3.3.7), we see that Z''_{n-1} , Z_n either vanish or are constant, while Z''_n vanishes. Also, the highest powers of z occur only in $\int^z\int^z Z_1$, so that we must have (cf. (3.3.1)):

$$A[T_1] \equiv 0 \quad . \tag{3.3.10}$$

Setting $\eta(r)$ equal to the trailing terms, the compatibility condition becomes

$$\begin{aligned}&(B[T_1]Z_1 + A[T_2]\int^z\int^z Z_2) + \\ &+ \sum_1^{n-2} (C[T_m]Z_m'' + B[T_{m+1}]Z_{m+1} + A[T_{m+2}]\int^z\int^z Z_{m+2}) = 0 \quad ,\end{aligned} \tag{3.3.11}$$

where T_1 is given by (3.3.1).

Clearly, an equation of the form

$$A[T_2] + \lambda\, B[T_1] = 0 \tag{3.3.12}$$

can be solved for T_2 if T_1 is known, for an arbitrary constant λ . Therefore if

$$Z_1 = \sum_0^{\nu} \alpha_m z^m \quad , \quad Z_2 = \sum_0^{\nu-2} \beta_m z^m \quad , \tag{3.3.13}$$

where ν is given by 3.3.7, we can have

$$A[T_2] \int^z\int^z Z_2 + B[T_1]Z_1 \equiv 0 \tag{3.3.14}$$

provided that

$$\frac{\alpha_{\nu-k}(\nu-k-1)(\nu-k)}{\beta_{\nu-k-2}} = \frac{\alpha_\nu(\nu-1)\nu}{\beta_{\nu-2}} = \lambda \ , \quad k = 0,\cdots,\nu-2 \quad , \tag{3.3.15}$$

or

$$\frac{\beta_{\nu-k-2}}{\beta_{\nu-k}} = \frac{(\nu-k-1)(\nu-k)}{(\nu-k+1)(\nu-k+2)} \cdot \frac{\alpha_{\nu-k}}{\alpha_{\nu-k+2}} \quad , \qquad k = 0,\cdots,\nu-2 \quad . \tag{3.3.16}$$

In the same way as above

$$A[T_3] + \lambda_1 B[T_2] + \mu_1 C[T_1] = 0 \tag{3.3.17}$$

can be solved for T_3 if T_1 and T_2 are known, for arbitrary constants λ_1, μ_1. Therefore if

$$Z_3 = \sum_0^{\nu-4} \gamma_m z^m \quad , \quad Z_1 \; , \; Z_2 \quad \text{as before,} \tag{3.3.18}$$

then we can have

$$A[T_3] \int^z \int^z Z_3 + B[T_2] Z_2 + C[T_1] Z_1'' \equiv 0 \tag{3.3.19}$$

provided that

$$\gamma_{\nu-k-4} = (\nu-k)(\nu-k-1)(\nu-k-2)(\nu-k-3) \frac{\alpha_{\nu-k}}{\mu_1}$$

$$= (\nu-k-2)(\nu-k-3) \frac{\beta_{\nu-k-2}}{\lambda_1} \quad , \qquad k = 0,\cdots,\nu-2 \quad . \tag{3.3.20}$$

These relationships are consistent with (3.3.15) if

$$\frac{\mu_1}{\lambda_1} = \lambda \quad . \tag{3.3.21}$$

It is thus possible to satisfy (3.3.14) and (3.3.19) by a suitable choice of polynomials Z_1, Z_2, Z_3. Thus if $Z_1 = z^4$, so that $\alpha_4 = 1$ and $\nu = 4$, and if we choose $\lambda = 12$, then from (3.3.15) we get $\beta_2 = 1$, so that $Z_2 = z^2$. Then from (3.3.20) we get $\gamma_o = \frac{24}{\mu_1} = \frac{2}{\lambda_1}$, where $\frac{\mu_1}{\lambda_1} = 12$,

and if we choose $\mu_1 = 24$, $\lambda_1 = 2$, then $\gamma_o = 1$. The compatibility condition then becomes

$$A[T_1]\frac{z^6}{30} + ([T_1] + \frac{A[T_2]}{12})z^4 + (12C[T_1] + B[T_2] + \frac{A[T_3]}{2})z^2$$

$$+ (2C[T_2] + B[T_3]) - \eta'(r) = 0 \quad ,$$

and all the coefficients of powers of z can be made to vanish by solving three ordinary differential equations and setting $\eta'(r) = 2C[T_2] + B[T_3]$.

In a similar manner, the identity

$$C[T_m]Z''_m + B[T_{m+1}]Z_{m+1} + A[T_{m+2}]\int^z\int^z Z_{m+2} \equiv 0 \tag{3.3.22}$$

may be satisfied provided that certain relationships exist among the coefficients of Z_{m+2} , Z_{m+1} , and Z_m . Thus the compatibility condition (3.3.9) may be satisfied by polynomials of arbitrary order.

Although we may determine several functions T_m and Z_m , not all of them give appropriate deformations for our specific problem. Since (3.1.5) must be satisfied, we must have, using (3.3.6),

$$-\sum_1^n \left(\frac{T'_m}{r}\right)' \int^z Z_m = 0 \quad , \quad z = 0, h_o \quad . \tag{3.3.23}$$

Therefore, since the functions are linearly independent, we must have

$$\left(\frac{T'_m}{r}\right)' = 0 \quad \text{for all} \quad m \ . \tag{3.3.24}$$

For arbitrary f this is impossible, unless $m = 1$. Then from (3.3.2) we get

$$\left(\frac{T_1'}{r}\right)' = \frac{\beta r}{f} \tag{3.3.25}$$

so we choose $\beta = 0$. Then

$$T_1 = \alpha\, r^2 \quad , \tag{3.3.26}$$

$$u = \alpha r\, Z_1 \quad , \quad v = -2\alpha \int Z_1 \quad , \tag{3.3.27}$$

and as shown previously, we must have Z_1 as contant, so we again have a deformation of the form (3.3.5). Therefore for our specific problem, the polynomial solutions for the homogeneous case are the only possible polynomial solutions of the inhomogeneous case.

If we define T_1 by (3.3.26), and set $Z_1 = 1$, then (3.2.6) becomes

$$\frac{1}{\Phi}\,\hat{\hat{p}} = [\alpha r\, f' - \frac{2\alpha}{f^2}] - \frac{\rho g z}{\Phi} + \eta(r) \quad , \tag{3.3.28}$$

and (3.2.8) becomes

$$[2\alpha(f + \frac{1}{f^2})' - \alpha(rf')'] - \eta'(r) = 0 \quad . \tag{3.3.29}$$

Therefore we have

$$\eta(r) = 2\alpha(f + \frac{1}{f^2}) - \alpha r f' + \gamma \quad , \tag{3.3.30}$$

$$\frac{1}{\Phi}\,\hat{p} = 2\,\alpha\, f - \frac{\rho g z}{\Phi} + \gamma \quad . \tag{3.3.31}$$

The boundary conditions (3.1.3) reduce to

$$-\gamma + \frac{\rho g z}{\Phi} = 0 \quad , \quad r = r_o \tag{3.3.32a}$$

$$0 = 0 \quad , \quad r = r_o \quad . \tag{3.3.32b}$$

Clearly this cannot be satisfied identically, so we must include other functions which are not polynomials.

3.4 Series Solutions.

Any function Z of the form $\cos \alpha z$ or $\sin \alpha z$ will reduce the condition (3.2.8) to an ordinary differential equation for T as a function of r. The remarks at the end of the previous section indicate that to satisfy the boundary conditions at $r = r_o$, we will have to use Fourier expansions of linear functions of z, on the interval $[0,h_o]$. The appropriate complete orthogonal set is $\{\cos n\pi \frac{z}{h_o} , \sin n\pi \frac{z}{h_o} , n \text{ odd}\}$. In order to satisfy the boundary conditions (3.1.5) on $z = 0,h_o$, we must express v in sine functions. We normalize the height of the cylinder by setting $h_o = 1$. Our perturbations are then given by

$$u = \alpha r + \sum_{1}^{\infty} n^2 \pi^2 \frac{T_n}{r} \cos n\pi z \quad , \quad n \quad \text{odd} \tag{3.4.1a}$$

$$v = -2\alpha z - \sum_{1}^{\infty} n\pi \frac{T_n'}{r} \sin n\pi z \quad , \quad n \quad \text{odd} \quad . \tag{3.4.1b}$$

The arbitrary function $\eta(r)$ which appears in (3.2.8) is sufficient to deal with the polynomial terms, but we must clearly satisfy (3.2.8) for each n independently. For each n, the compatibility condition (3.2.8) takes the form

$$n^2\pi^2 \left[\left(\left\{f + \frac{1}{f^2}\right\} \frac{T_n'}{r}\right)' - (rf')' \frac{T_n}{r}\right] \cos n\pi z - n^4\pi^4 \frac{T_n}{rf^2} \cos n\pi z$$

$$- \left[\frac{1}{r} \left(rf \left\{\frac{T_n'}{r}\right\}'\right)'\right]' n^4\pi^4 \cos n\pi z = 0 \quad . \qquad (3.4.2)$$

For a given n , set

$$s = \frac{n^2\pi^2 r^2}{4} \quad , \qquad (3.4.3)$$

and let * denote differentiation with respect to s . After some reduction, we find that (3.4.2) may be written in the particular form

$$s[sf\ T^{**}]^{**} - s[(f + \frac{1}{f^2})T^*]^* + [(sf^*)^* + \frac{1}{f^2}]T = 0 \quad . \qquad (3.4.4)$$

This equation is irreducible for arbitrary f , although if we define operators F, G by

$$F \equiv sf \frac{d^2}{ds^2} - f \quad , \qquad (3.4.5a)$$

$$G \equiv sf \frac{d^2}{ds^2} - \frac{1}{f^2} \quad , \qquad (3.4.5b)$$

we may write

$$F[G[T]] - G[F[T]] = \frac{sf}{T} [(f - \frac{1}{f^2})^* T^2]^* \quad , \qquad (3.4.6a)$$

$$F[G[T]] + G[F[T]] = [-2f(sf^*)^* - sf(f + \frac{1}{f^2})^{**}]T \quad . \qquad (3.4.6b)$$

In the homogeneous case, when $f = h_1$, a constant, the expressions on the right vanish, and we find that

$$F[G[T]] = G[F[T]] = 0 \quad . \qquad (3.4.7)$$

The equation (3.4.4) may be written

$$F[G[T]] + G[F[T]] = 0 \qquad (3.4.8)$$

in the homogeneous case, and since the operators commute, we obtain two linearly independent non-logarithmic solutions from (3.4.7), namely

$$T_1 = \sqrt{s}\, I_1\, (2\sqrt{s}) \quad , \qquad (3.4.9a)$$

$$T_2 = \sqrt{\frac{s}{h_1^3}}\, I_1 (2 \sqrt{\frac{s}{h_1^3}}) \quad , \qquad (3.4.9b)$$

where I_1 is the Bessel function. The general solution is

$$T = \alpha_1\, T_1 + \alpha_2\, T_2 \quad , \qquad (3.4.10)$$

where α_1 and α_2 are constants. This form leads to Vaughan's solution.

We will assume for the moment that (3.4.4) has two linearly independent solutions for each harmonic (each n) , and will show how the boundary conditions may be satisfied.

3.5 Satisfaction of Boundary Conditions.

The conditions (3.1.5) on the horizontal ends have already been satisfied by our choice of the form (3.4.1). We require also, by (3.1.4), that $u(0,z)$ vanish for all z . Thus from (3.4.1) we must have

$$\sum_1^\infty [\lim_{r\to 0} \frac{T_n}{r}]\, n^2\pi^2 \cos n\pi z = 0 \quad , \quad n \text{ odd} \quad , \qquad (3.5.1)$$

and therefore we must have

$$\lim_{r\to 0} \frac{T_n}{r} = 0 \quad , \quad n \quad \text{odd} \quad . \tag{3.5.2}$$

When we discuss solutions in detail, we will point out that for small r the solutions behave like the solutions (3.4.9), and satisfy this condition.

With u,v defined by (3.4.1) and η defined by (3.3.30), the condition (3.2.6) becomes

$$\frac{1}{\Phi}\hat{p} = 2\alpha f - \frac{\rho g z}{\Phi} + \gamma + \sum_1^\infty [f' \frac{T_n}{r} - \frac{T_n'}{rf^2}] n^2\pi^2 \cos n\pi z$$

$$+ \sum_1^\infty \frac{1}{r}[rf\,(\frac{T_n'}{r})']' \cos n\pi z \quad , \quad n \quad \text{odd} \tag{3.5.3}$$

and $\dfrac{\partial u}{\partial r}$ becomes

$$\frac{\partial u}{\partial r} = \alpha + \sum_1^\infty [\frac{T_n}{r}]' \, n^2\pi^2 \cos n\pi z \quad , \tag{3.5.4}$$

so that condition (3.1.3a) becomes

$$\frac{\rho g z}{\Phi} - \gamma + \sum_1^\infty [2f \frac{T_n'}{r} - 2f \frac{T_n}{r^2} - f' \frac{T_n}{r} + \frac{T_n'}{rf^2}] \, n^2\pi^2 \cos n\pi z$$

$$- \sum_1^\infty \frac{1}{r}[rf\,(\frac{T_n'}{r})']' \cos n\pi z = 0 \quad , \quad n \text{ odd} \, , \, r = r_o \quad . \tag{3.5.5}$$

The only linear function of z that can be expanded on $[0,1]$ by odd cosine terms alone is $(z - \frac{1}{2})$ and its multiples. Therefore we set our arbitrary constant

$$\gamma = \frac{\rho g}{2\Phi} \quad . \tag{3.5.6}$$

The coefficient of $\cos n\pi z$ in the expansion of $(z - \frac{1}{2})$ is $\frac{-4}{n^2\pi^2}$, so at $r = r_o$ we must have

$$[2f \frac{T_n'}{r} - 2f \frac{T_n}{r^2} - f' \frac{T_n}{r} + \frac{T_n'}{rf^2}] n^2\pi^2 - \frac{1}{r} [rf (\frac{T_n'}{r})']' = \frac{4\rho g}{\Phi n^2 \pi^2} , \quad r = r_o \quad . \tag{3.5.7}$$

In terms of the variable s, this becomes

$$[sf T^{**}]^* - [2f + \frac{1}{f^2}]T^* + [f^* + \frac{f}{s}]T = \frac{-8\rho g}{\Phi n^6 \pi^6} , \quad s = s_o \quad . \tag{3.5.8}$$

In order to satisfy (3.1.3b), we must have

$$\sum_1^\infty - \frac{T_n}{r} n^3\pi^3 \sin n\pi z - \sum_1^\infty [\frac{T_n'}{r}]' n\pi \sin n\pi z = 0 \quad , \tag{3.5.9}$$

or in terms of s,

$$s T^{**} + T = 0 \quad , \quad s = s_o \quad . \tag{3.5.10}$$

For each n we obtain two linearly independent solutions of (3.4.4), and hence a general solution

$$T = \alpha_1 T_1 + \alpha_2 T_2 \quad . \tag{3.5.11}$$

The equations (3.5.8) and (3.5.10) then become two simultaneous equations in α_1, α_2. Thus the solution (3.5.11) is unique, unless the determinant of the system vanishes. We will investigate the behaviour of the determinant when we give our numerical results in the next two sections.

3.6 The Power-Series Approach.

We were unable to determine the solutions of (3.4.4) in closed form. Atempts to derive explicit expressions for T by means of power-series, by expanding f in an arbitrary power-series, also achieved no success. We thus decided to investigate solutions corresponding to an f of simple form. We chose

$$f = h_1(1 + \epsilon r^2) \quad , \; |\epsilon r^2| < 1 \quad . \tag{3.6.1}$$

Our solutions for the homogeneous case are then obtained by setting ϵ to be zero. In terms of s, the function f has the form

$$f = h_1(1 + Ms) \quad , \tag{3.6.2}$$

where

$$M = \frac{4\epsilon}{n^2\pi^2} \quad . \tag{3.6.3}$$

The laminations are characterized by the values that ϵ assumes in (3.6.1). For each such value of ϵ, we obtain one value of M for each harmonic considered, i.e. for each n, n odd.

The solutions have to be calculated on an r - interval $[0,r_o]$ which corresponds to an s - interval $[0 , \frac{n^2\pi^2 r_o^2}{4}]$. Thus the solutions corresponding to higher harmonics must be calculated over longer intervals.

Having chosen the form of f, the equation (3.4.4) becomes

$$s^2[1+Ms]T^{iv} + 2s[1+2Ms]T^{***} + s[2M - 1 - Ms - \frac{L}{(1+Ms)^2}]T^{**}$$

$$- s[M - \frac{2LM}{(1+Ms)^2}]T^{*} + [M + \frac{L}{(1+Ms)^2}]T = 0 \quad , \qquad (3.6.4)$$

where we have defined a constant

$$L = \frac{1}{h_1^3} \qquad (3.6.5)$$

which is less than unity unless there is no compression.

We also write down the explicit form of the boundary conditions. If we define constants

$$s_1 = \frac{\pi^2 r_o^2}{4} \quad , \quad M_1 = \frac{4\epsilon}{\pi^2} \quad , \quad \sigma_1 = 1 + M_1 s_1 \quad , \qquad (3.6.6)$$

then for each harmonic considered, the system (3.5.8) and (3.5.10) reduces to

$$n^2 s_1 [1+\sigma_1]V^{***} + [1+2\sigma_1]V^{**} - [2 + 2\sigma_1 + \frac{L}{(1+\sigma_1)^2}]V^{*}$$

$$+ [\frac{1+2\sigma_1}{n^2 s_1}]V = - \frac{1}{n^6} \quad , \qquad (3.6.7a)$$

$$n^2 s_1 V^{**} + V = 0 \quad , \qquad (3.6.7b)$$

where we have eliminated the unknown constants ρ, g, Φ by setting

$$V = \frac{\Phi \pi^6 h_1}{8 \rho g} T \quad . \qquad (3.6.8)$$

Clearly, V is a solution of the form (3.5.11).

If we choose an M_1 which causes σ_1 to vanish, then the system (3.6.7) reduces to

$$- L V^* = 0 \quad , \tag{3.6.9a}$$

$$n^2 s_1 V^{**} + V = 0 \quad . \tag{3.6.9b}$$

The equation (3.6.4) has a regular singular point at the origin, and can be solved by the appropriate power-series method. We set

$$T = \sum_0^\infty \frac{a_m}{m!} s^{m+c} \quad . \tag{3.6.10}$$

If we substitute in (3.6.4), we get

$$\begin{aligned}
&\sum_0^\infty [M^4s^6 + 4M^3s^5 + 6M^2s^4 + 4Ms^3 + s^2][m+c][m+c-1][m+c-2][m+c-3] \frac{a_m}{m!} s^{m+c-4} \\
&+ \sum_0^\infty [4M^4s^5 + 14M^3s^4 + 18M^2s^3 + 10Ms^2 + 2s][m+c][m+c-1][m+c-2] \frac{a_m}{m!} s^{m+c-3} \\
&+ \sum_0^\infty [2M^4s^4 + 6M^3s^3 + 6M^2s^2 + 2Ms][m+c][m+c-1] \frac{a_m}{m!} s^{m+c-2} \\
&- \sum_0^\infty [M^4s^5 + 4M^3s^4 + 6M^2s^3 + 4Ms^2 + s][m+c][m+c-1] \frac{a_m}{m!} s^{m+c-2} \\
&- \sum_0^\infty [LMs^2 + Ls][m+c][m+c-1] \frac{a_m}{m!} s^{m+c-2} \\
&- \sum_0^\infty [M^4s^4 + 3M^3s^3 + 3M^2s^2 + Ms][m+c] \frac{a_m}{m!} s^{m+c-1} \\
&+ \sum_0^\infty 2LMs[m+c] \frac{a_m}{m!} s^{m+c-1} \quad + \quad \sum_0^\infty [M^4s^3 + 3M^3s^2 + 3M^2s + M] \frac{a_m}{m!} s^{m+c} \\
&+ \sum_0^\infty L[Ms+1] \frac{a_m}{m!} s^{m+c} \quad = \quad 0 \; .
\end{aligned} \tag{3.6.11}$$

The indicial equation is found to be

$$c[c-1][c-2][c-3]\, a_o\, s^{c-2} + 2c[c-1][c-2]\, a_o\, s^{c-2} = 0 \,, \qquad (3.6.12)$$

$$c = 0,1,1,2 \quad . \qquad (3.6.13)$$

Set $c = 0$ and gather terms in powers of s. After rearranging the summations, the equation (3.6.11) may be written

$$- \sum_3^\infty M^4[m-1][m-2]^2[m-4] \frac{a_{m-3}}{(m-1)!} s^m + \sum_2^\infty M^3[m-1][m-3][M(m-2)^2(m-3)-(4m-5)] \frac{a_{m-2}}{(m-1)!} s^m$$

$$+ \sum_1^\infty [2M^3(m-1)(m-2)^2(2m-3) - 3M^2(2m-1)(m-2) + LM] \frac{a_{m-1}}{(m-1)!} s^m$$

$$+ \sum_0^\infty [6M^2 m(m-1)^3 - M(4m+1)(m-1) - LMm(m-3) + L] \frac{a_m}{m!} s^m$$

$$+ \sum_1^\infty [2Mm(2m-1) - (1+L)] \frac{a_{m+1}}{(m-1)!} s^m + \sum_1^\infty [m+1] \frac{a_{m+2}}{(m-1)!} s^m = 0 \quad . \qquad (3.6.14)$$

The partial and full recurrence relations are:

$m = 0$: $$[M+L]\, a_o = 0 \quad , \quad a_o = 0 \qquad (3.6.15a)$$

$m = 1$: $$2a_3 + [2M - (1+L)]a_2 + L[2M+1]a_1 = 0 \qquad (3.6.15b)$$

$m = 2$: $$3a_4 + [12M - (1+L)]a_3 + [12M^2-9M+2LM+L] \frac{a_2}{2} + LMa_1 = 0 \qquad (3.6.15c)$$

$m \geq 3$: $$a_{m+2} = M^4[m-1][m-2]^2[m-4] \frac{a_{m-3}}{m+1}$$

$$- M^3[m-1][m-3][M(m-2)^2(m-3)-(4m-5)] \frac{a_{m-2}}{m+1}$$

$$- [2M^3(m-1)(m-2)^2(2m-3) - 3M^2(2m-1)(m-2)+LM] \frac{a_{m-1}}{m+1}$$

$$- [6M^2m(m-1)^3 - M(4m+1)(m-1) - LMm(m-3) + L] \frac{a_m}{m[m+1]}$$

$$- [2Mm(2m-1) - (1+L)] \frac{a_{m+1}}{m+1} \quad . \tag{3.6.16}$$

In the homogeneous case, $M = 0$, these reduce to:

$$m = 0 \quad : \qquad La_o = 0 \quad , \quad a_o = 0 \quad , \tag{3.6.17a}$$

$$m = 1 \quad : \qquad 2a_3 - [1+L]a_2 + La_1 = 0 \quad , \tag{3.6.17b}$$

$$m \geq 2 \quad : \qquad a_{m+2} = [1+L] \frac{a_{m+1}}{m+1} - \frac{L\, a_m}{m[m+1]} \quad . \tag{3.6.18}$$

Clearly, from (3.6.15b), any two of a_1 , a_2 , a_3 may be chosen arbitrarily, so we can obtain two linearly independent solutions. Our first solution, which we call T_1 , is obtained by choosing

$$a_1 = 1 \quad , \quad a_2 = 1 \quad . \tag{3.6.19}$$

This solution corresponds to the solution (3.4.9a). The second solution, T_2 , corresponding to (3.4.9b), is obtained by choosing

$$a_1 = 1 \quad , \quad a_2 = L \quad . \tag{3.6.20}$$

At the origin, the inhomogeneous and corresponding homogeneous solutions match in value, slope, and concavity. The boundary condition (3.1.4) is thus satisfied, as indicated in section 3.5.

A program was written to compute the first fifty a_m's for various values of L and M . There was some oscillation in signs of the a_m's for larger values of M , but their absolute values decreased extremely rapidly,

and a further division by $m!$ to obtain the coefficients of s^m in (3.6.10) ensured that the last thirty or more a_m's had negligible effect on solutions.

The power-series method was an inefficient use of computer time. It was only used for two specific purposes. It was used to calculate solutions and their first three derivatives at a small s-value. These results were then used an input for a faster and more efficient program. The method was also used to calculate the two solutions corresponding to a particular M - value, $M = -\dfrac{4}{r_o^2\,\pi^2}$, which caused the function σ_1 to vanish. This particular result was used in studying the behaviour of the determinant of the system (3.6.9).

3.7 The Numerical Solution.

The equation (3.6.4) was converted to a system of four first-order ordinary differential equations. Setting

$$\sigma = 1 + Ms \,, \tag{3.7.1}$$

and dividing (3.6.4) by $s^2\sigma$, we are led to the system

$$\begin{bmatrix} T \\ T^* \\ T^{**} \\ T^{***} \end{bmatrix}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{1}{s^2\sigma}\left[M + \dfrac{L}{\sigma^2}\right] & \dfrac{M}{s\sigma}\left[1 - \dfrac{2L}{\sigma^3}\right] & \dfrac{1}{s\sigma}\left[\sigma + \dfrac{L}{\sigma^2} - 2M\right] & -\dfrac{2}{s\sigma}\left[1+2Ms\right] \end{bmatrix} \begin{bmatrix} T \\ T^* \\ T^{**} \\ T^{***} \end{bmatrix} \tag{3.7.2}$$

This system was solved on an IBM 360-67 installation using the "DASCRU" program from the IMSL library. The program uses a modified Runge-Kutta algorithm. It is the fastest program available for this purpose and has an accuracy of approximately machine-word-length. Required input includes the order of the system, an initial-value vector, and the function which determines the final row of the matrix. Output is a solution vector at a specified s - value.

The "DASCRU" program could not handle the system (3.7.2) when s or σ vanished. This is why the initial-value vector was calculated by power-series.

In order to evaluate boundary conditions, it was necessary to choose a specific value of r_o. We chose

$$r_o = \frac{\sqrt{10}}{\pi} . \tag{3.7.3}$$

In terms of s, the interval over which solutions had to be calculated became, for each n, $[0,2.5n^2]$. The constant s_1 in (3.6.7) became 2.5.

For each choice of M and L, the power-series method was used to calculate an intial - value vector for each linearly independent solution, for each value of n. Each initial-value vector was used as input to "DASCRU", which gave solutions at ten equal intervals outward from $.25n^2$ to $2.5n^2$, using the output from each step as input for the next. The solutions are tabulated in Appendix 2.

Appendix 2 also includes various intermediate results, mentioned below, leading to the final form of the perturbations.

The solutions and their first two derivatives were positive everywhere, in fact the second derivative had positive minimum. Thus there is a strong indication that the general equation (3.4.4) is non-oscillatory. The solutions for negative M are apparently bounded below by the homogeneous solutions (the Bessel functions), while the solutions for positive M are bounded above by them.

The boundary values were used as input for a program designed to solve the system (3.6.7), as indicated in section 3.5. For all M - values for which σ_1 did not vanish, the determinant was positive and became very large for the higher harmonics. We were thus able to obtain unique solutions of the form (3.5.11) for the significant harmonics. The solutions for $\sigma_1 = 0$ were calculated by power-series. The determinant of the system was then found to be a three-digit number representing the difference of two ten-digit numbers. Therefore, within the limits of machine accuracy, the determinant of the system (3.6.9) vanished. We were thus unable to obtain a "limiting" solution of the form (3.5.11).

The unique solutions obtained above were then converted into a form suitable for substitution in (3.4.1). The series contributions to the perturbations were then calculated, to three or four harmonics, i.e. to within 1.5% . The final results included various constant scaling factors, but comparisons could be made.

The horizontal perturbations were up to 45% smaller for $M = -2$, than for the homogeneous case, while those corresponding to $M = +.2$ were up to 45% greater. There were also significant, though not monotonic, effects on the vertical components. In general, a negative M - value gave

less vertical perturbation as well.

The polynomial terms in (3.4.1) represent the first-order form of a universal solution of the type (2.3.1). Hence it may be assumed that these terms are included in the initial compression, so that the actual perturbation is obtained by setting $\alpha = 0$ in (3.4.1).

We have thus shown that a laminated metric of the type (2.3.13) prohibits any other deformation of polynomial type from being superimposed on a solution of the type (2.3.1), that is, a simple compression.

We have also shown that the laminations have a considerable effect on the magnitude of the perturbations due to gravity.

APPENDIX I

Thermodynamics and Symmetry of the Constitutive Equation

This is an outline of the work of Rivlin [3] and Ericksen and Rivlin [1], in which they developed the simple form of the constitutive equation (I.3.4) for an isotropic solid under an isothermal deformation.

The material time-derivative of the position vector $\underline{x}$ is denoted by $v^i = \dot{x}^i$. We deal with rectangular Cartesian systems.

Denote absolute temperature by K, surface forces by t^i, body forces by f^i, the rate of heat generation by r, heat flux by q, original density by ρ_o, internal energy by U, and entropy by S, all quantities defined per unit mass or per unit undeformed area.

The balance-of-energy equation then becomes

$$\int_V \rho_o(r+f^i v_i)dV + \int_A (t^i v_i - q)dA = \int_V \rho_o(\dot{U}+\dot{v}^i v_i)dV \quad , \tag{AI.1}$$

while the Clausius-Duhem inequality is given by

$$\int_V \rho_o \dot{S}\, dV \geq \int_V \frac{\rho_o}{K} r\, dV - \int_A \frac{q}{K} dA \quad . \tag{AI.2}$$

If in an elementary tetrahedron we let π^{iA} be the force vector acting on the surface whose normal is n_A, and q^A be the heat flux through that surface, then in the limiting case as the volume decreases to zero, we must have

$$t^i v_i - q = \pi^{iA} n_A v_i - q^A n_A \quad . \tag{AI.3}$$

If we impose a constant translational velocity $\underline{c}$, we get

$$t^i v_i + t^i c_i - q = \pi^{iA} n_A v_i + \pi^{iA} n_A c_i - q^A n_A \tag{AI.4}$$

and therefore we must have

$$t^i = \pi^{iA} n_A \tag{AI.5a}$$

$$q = q^A n_A \quad . \tag{AI.5b}$$

Putting these results into (AI.1), we obtain, after some reduction,

$$(\rho_o f^i + \pi^{iA}{}_{|A} - \rho_o \dot{v}^i) v_i + \rho_o r + \pi^{iA} v_{i|A} - q^A{}_{|A} = \rho_o \dot{U} \quad . \tag{AI.6}$$

Again considering the effect of an imposed velocity $\underline{c}$, we conclude

$$\rho_o f^i + \pi^{iA}{}_{|A} = \rho_o \dot{v}^i \quad , \tag{AI.7a}$$

$$\pi^{iA} v_{i|A} + \rho_o r - q^A{}_{|A} = \rho_o \dot{U} \quad . \tag{AI.7b}$$

Using the divergence theorem, (AI.2) becomes

$$\rho_o \dot{S} \geq \frac{\rho_o r}{K} - \frac{q^A{}_{|A}}{K} + \frac{q^A K,_A}{K^2} \quad , \tag{AI.8}$$

and using (AI.7b), we find

$$\rho_o (\dot{U} - K\dot{S}) \leq \pi^{iA} v_{i|A} - \frac{q^A K,_A}{K} \quad . \tag{AI.9}$$

Define the Helmholtz free-energy by

$$A = U - KS \quad . \tag{AI.10}$$

Since the deformation is isothermal, $\dot{K}$ vanishes, and in (AI.9) we have

$$\rho_o \dot{A} + \frac{q^A K,_A}{K} \leq \pi^{iA} \; v_{i|A} \quad . \tag{AI.11}$$

The energy A depends explicity only on the deformation gradients, and not on the time, so (AI.11) becomes

$$(\rho_o \frac{\partial A}{\partial x^j,_A} - \pi^{iA} \; \delta_{ij}) v^j{}_{|A} \leq - \frac{q^A}{K} K,_A \quad . \tag{AI.12}$$

To satisfy this identically for all deformations, we must have

$$q^A K,_A \leq 0 \quad , \tag{AI.13a}$$

$$\rho_o \frac{\partial A}{\partial x^j,_A} = \pi^{iA} \; \delta_{ij} \quad . \tag{AI.13b}$$

The stress tensor τ^{ij} is related to the π's by

$$\tau^{ij} = \frac{\pi^{iA} x^j,_A}{\det F} \quad , \tag{AI.14}$$

where F is the deformation gradient matrix. Thus

$$\tau^{ij} = \rho \; \delta^{mi} x^j,_A \frac{\partial A}{\partial x^m,_A} \quad , \tag{AI.15}$$

where ρ is density in the deformed state.

By the principle of <u>material frame indifference</u>, the energy A must be invariant under all orthogonal changes of frame. It may then be shown that A must depend explicitly only the the C_{PQ} , defined by

$$C_{PQ} = x^i,_P \; x^j,_Q \; \delta_{ij} \quad . \tag{AI.16}$$

Equation (AI.15) then reduces to

$$\tau^{ij} = \rho(x^i{}_{,Q}\, x^j{}_{,P} + x^i{}_{,P}\, x^j{}_{,Q}) \frac{\partial A}{\partial C_{PQ}} \quad . \tag{AI.17}$$

This is a standard result for all bodies. It may be simplified further by noting that τ is invariant under transformations of the isotropy group.

If the body is transversely isotropic, an element of the group is

$$((H^r_s)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos w & \sin w \\ 0 & -\sin w & \cos w \end{pmatrix} \quad , \tag{AI.18}$$

and if we define

$$\tilde{C}_{PL} = C_{PQ}\, H^Q_L \tag{AI.19}$$

then we must have

$$\frac{dA}{dw} = \frac{\partial A}{\partial \tilde{C}_{PQ}} \frac{\partial \tilde{C}_{PQ}}{\partial w} = 0 \quad . \tag{AI.20}$$

This identity can only hold if there are certain relationships among the components of C, so that A depends explicitly only on certain functions of the components of C. These functions also satisfy certain identities involving powers of C and powers of B, where

$$B^{rs} = x^r{}_{,A}\, x^s{}_{,P}\, \delta^{AP} \quad . \tag{AI.21}$$

In particular, for an isotropic material, the group element in (AI.18) is any member of the group $SO(3)$, and the identity (AI.20) requires that A depend only on the three invariants of B , so that the final form of (AI.17) becomes

$$\tau = \eta_0 I + \eta_1 B + \eta_{-1} B^{-1} \quad , \tag{AI.22}$$

where the η's are functions of the three invariants. Since B is non-singular and satisfies its characteristic polynomial,

$$B^2 - I_1 B + I_2 I - I_3 B^{-1} = 0 \quad , \tag{AI.23}$$

another form of the constitutive equation is

$$\tau = \xi_0 I + \xi_1 B + \xi_2 B^2 \quad . \tag{AI.24}$$

APPENDIX 2

Numerical Tables

In this appendix we give a sample of the numerical results which illustrate the methods and conclusions of Chapter 3. In particular, we give horizontal and vertical components of the displacements of representative points in a laminated, compressed cylinder perturbed by gravity. The quantities calculated varied smoothly with respect to all parameters, and a condensed sample is sufficient to indicate the behaviour of solutions.

The results given are the series contributions to the displacements (3.4.1), up to the fourth harmonic. The results are also scaled, and are thus related to the u,v of (3.4.1) by

$$u_{given} = \frac{\Phi\pi^4 h_1}{8\rho g}(u-\alpha r)$$

$$v_{given} = \frac{\Phi\pi^4 h_1}{8\rho g}(v+2\alpha z) \quad .$$

The representative points lie on two concentric cylindrical shells, an interior one of radius $\frac{\sqrt{3}}{\pi}$, and the outer surface, radius $\frac{\sqrt{10}}{\pi}$. For each harmonic, the corresponding s - values are $.75n^2$ and $2.5n^2$.

This appendix is in three sections, each one corresponding to a particular lamination characterized by setting $M_1 = -.2$, 0, $+.2$ in (3.6.6). The second section, $M_1 = 0$, represents the homogeneous case. The constant L in (3.6.5) is assumed to be $.729$, representing a compression $z = .9Z$.

In each section, the first page of tables gives the solutions T_1 and T_2 of (3.7.2) for each harmonic. For the case $M_1 = 0$, the required solutions and derivatives were obtained from Bessel-function subroutines, which were simpler than DASCRU.

The second page includes the determinant Δ of the system (3.6.7), and the constants determined by it, for each harmonic. The unique solution (3.5.11) thus obtained is multiplied by n^2 and listed, along with its derivative. These are the actual coefficients of the sine and cosine terms, except for the scaling factor $\dfrac{8\rho g}{\Phi \pi^4 h_1}$.

Finally, the horizontal and vertical components of displacement are given, for various z - values. A diagram showing these displacements is also included.

A2.1 $M_1 = -.2$

DASCRU Output

$s = .75\ n^2$

	n=1	3	5	7
T_1	1.103E 00	8.825E 01	4.179E 03	1.748E 05
T_1^*	2.056E 00	4.124E 01	1.120E 01	3.279E 04
T_1^{**}	1.929E 00	1.651E 01	2.752E 02	5.789E 03
T_1^{***}	1.777E 00	5.911E 00	6.286E 01	9.693E 02
T_2	1.002E 00	5.122E 01	1.463E 03	3.551E 04
T_2^*	1.747E 00	2.111E 01	3.407E 02	5.658E 03
T_2^{**}	1.336E 00	7.287E 00	7.157E 01	8.313E 02
T_2^{***}	1.154E 00	2.251E 00	1.384E 01	1.134E 02

$s = 2.5\ n^2$

	n=1	3	5	7
T_1	1.076E 01	4.166E 04	1.267E 08	3.794E 11
T_1^*	1.249E 01	1.767E 04	3.454E 07	7.659E 10
T_1^{**}	1.592E 01	8.228E 03	1.001E 07	1.613E 10
T_1^{***}	2.706E 01	4.250E 03	3.080E 06	3.539E 09
T_2	8.118E 00	1.339E 04	1.589E 07	1.117E 10
T_2^*	8.661E 00	5.405E 03	4.227E 06	2.214E 09
T_2^{**}	1.042E 01	2.455E 03	1.211E 06	4.623E 08
T_2^{***}	1.761E 01	1.256E 03	3.709E 05	1.011E 08

	n=1	3	5	7
Δ	8.650E 01	1.280E 08	1.589E 14	1.899E 20
α_1	-3.950E -01	-7.354E -07	-3.688E -11	-3.034E -15
α_2	5.847E -01	2.431E -06	3.029E -10	1.054E -13

$s = .75\ n^2$

	n=1	3	5	7
T	2.724E -01	9.729E -04	1.311E -05	2.855E -07
$.5n\pi T^*$	-3.288E -01	-8.897E -04	-1.216E -05	-2.676E -07

$s = 2.5\ n^2$

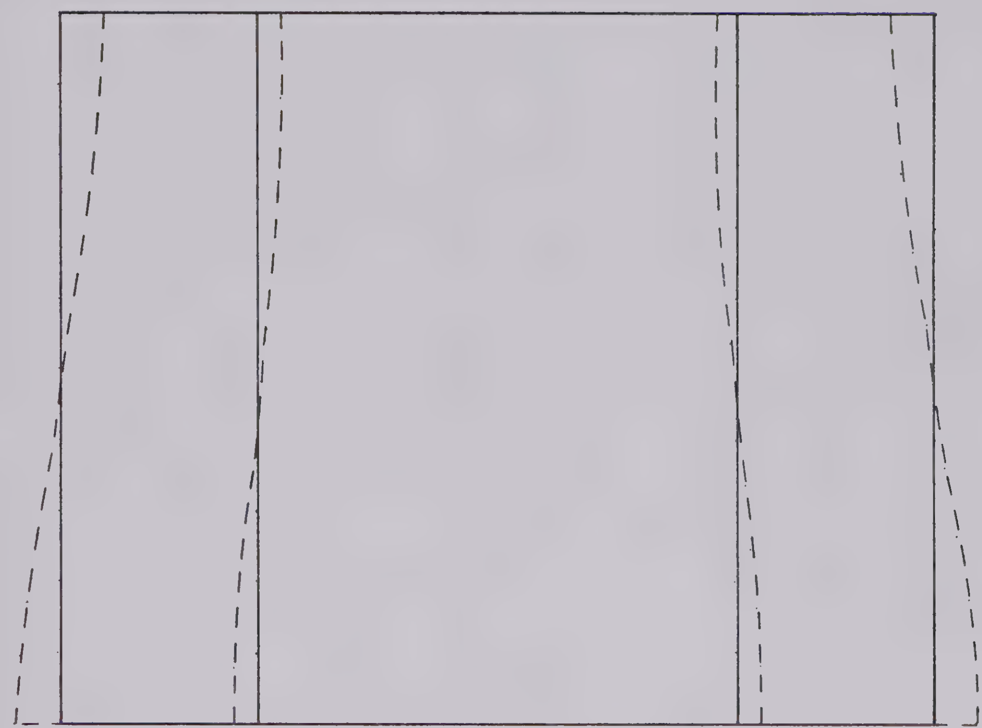
	n=1	3	5	7
T	4.923E -01	1.710E -02	3.549E -03	1.268E -03
$.5n\pi T^*$	-2.038E -01	-6.089E -03	-1.322E -03	-4.845E -04

$s = .75\ n^2$

	u	v
z=0	2.734E -01	0.0
.10	2.597E -01	-1.023E -01
.20	2.201E -01	-1.941E -01
.30	1.592E -01	-2.663E -01
.40	8.341E -02	-3.122E -01
.50	0.0	-3.280E -01
.60	-8.341E -02	-3.122E -01
.70	-1.592E -01	-2.663E -01
.80	-2.201E -01	-1.941E -01
.90	-2.597E -01	-1.023E -01
1.0	-2.734E -01	0.0

$s = 2.5\ n^2$

	u	v
z=0	5.142E -01	0.0
.10	4.775E -01	-6.960E -02
.20	3.890E -01	-1.251E -01
.30	2.743E -01	-1.656E -01
.40	1.408E -01	-1.905E -01
.50	0.0	-1.985E -01
.60	-1.408E -01	-1.905E -01
.70	-2.743E -01	-1.656E -01
.80	-3.890E -01	-1.251E -01
.90	-4.775E -01	-6.960E -01
1.0	-5.142E -01	0.0



PERTURBATION of a LAMINATED CYLINDER

$M_1 = -.2$

A2.2 $M_1 = 0$

Output of Bessel Function Subroutines

$s = .75\ n^2$

	n=1	3	5	7
T_1	1.069E 00	7.574E 01	3.234E 03	1.239E 05
T_1^*	1.903E 00	3.247E 01	7.942E 02	2.135E 04
T_2	9.746E -01	4.419E 01	1.146E 03	2.654E 04
T_2^*	1.626E 00	1.653E 01	2.430E 02	3.933E 03

$s = 2.5\ n^2$

	n-1	3	5	7
T_1	7.235E 00	7.769E 03	5.695E 06	3.788E 09
T_1^*	5.572E 00	1.732E 03	7.443E 05	3.503E 08
T_1^{**}	2.894E 00	3.453E 02	9.112E 04	3.092E 07
T_1^{***}	1.071E 00	6.162E 01	1.045E 04	2.607E 06
T_2	5.585E 00	2.442E 03	7.124E 05	1.882E 08
T_2^*	3.842E 00	4.695E 02	7.996E 04	1.492E 07
T_2^{**}	1.629E 00	7.911E 01	8.310E 03	1.120E 06
T_2^{***}	4.687E -01	1.170E 01	7.997E 02	7.965E 04

	n=1	3	5	7
Δ	1.377E 01	1.721E 06	2.188E 11	2.741E 16
α_1	-7.013E -01	-3.365E -06	-3.603E -10	-1.009E -13
α_2	1.051E 00	1.238E -05	3.332E -09	2.349E -12

$s = .75\ n^2$

	n=1	3	5	7
T	4.982E -01	4.774E -03	1.203E -04	4.429E -06
$.5n\pi T^*$	-5.880E -01	-4.050E -03	-1.028E -04	-3.817E -06

$s = 2.5\ n^2$

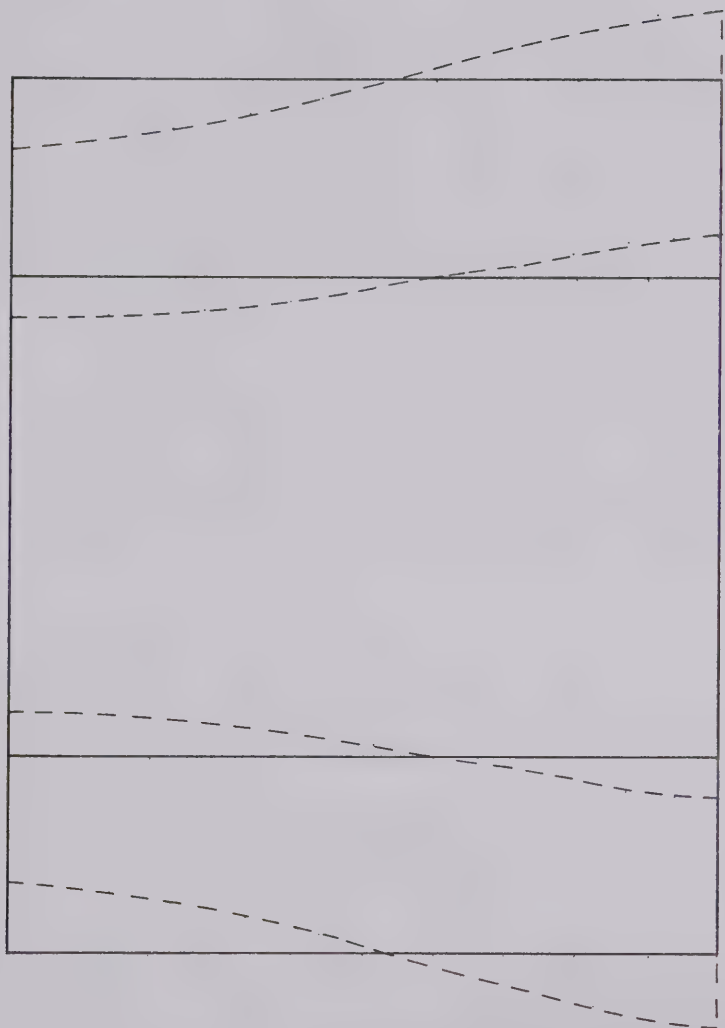
	n=1	3	5	7
T	7.901E -01	3.663E -02	7.988E -03	2.916E -03
$.5n\pi T^*$	-2.036E -01	5.353E -04	3.470E -04	1.586E -04

$s = .75\ n^2$

	u	v
z=0	5.031E -01	0.0
.10	4.766E -01	-1.851E -01
.20	4.015E -01	-3.495E -01
.30	2.883E -01	-4.769E -01
.40	1.502E -01	-5.569E -01
.50	0.0	-5.841E -01
.60	-1.502E -01	-5.569E -01
.70	-2.883E -01	-4.769E -01
.80	-4.015E -01	-3.495E -01
.90	-4.766E -01	-1.851E -01
1.0	-5.031E -01	0.0

$s = 2.5\ n^2$

	u	v
z=0	8.376E -01	0.0
.10	7.712E -01	-6.201E -02
.20	6.190E -01	-1.193E -01
.30	4.323E -01	-1.648E -01
.40	2.201E -01	-1.939E -01
.50	0.0	-2.039E -01
.60	-2.201E -01	-1.939E -01
.70	-4.323E -01	-1.648E -01
.80	-6.190E -01	-1.193E -01
.90	-7.712E -01	-6.201E -02
1.0	-8.376E -01	0.0



PERTURBATION of a HOMOGENEOUS CYLINDER

$M_1 = 0$

A2.3 $M_1 = +.2$

DASCRU Output

$s = .75\ n^2$

	n=1	3	5	7
T_1	1.042E 00	6.716E 01	2.681E 03	9.824E 04
T_1^*	1.792E 00	2.711E 01	6.255E 02	1.623E 04
T_1^{**}	1.113E 00	8.493E 00	1.273E 02	2.443E 03
T_1^{***}	1.533E -01	2.055E 00	2.260E 01	3.356E 02
T_2	9.533E -01	3.940E 01	9.655E 02	2.240E 04
T_2^*	1.540E 00	1.378E 01	1.954E 02	3.275E 03
T_2^{**}	7.119E -01	3.537E 00	3.398E 01	4.373E 02
T_2^{***}	-1.542E -02	6.674E -01	5.151E 00	5.389E 01

$s = 2.5\ n^2$

	n=1	3	5	7
T_1	6.034E 00	4.684E 03	3.087E 06	1.969E 09
T_1^*	4.011E 00	9.467E 02	3.824E 05	1.759E 08
T_1^{**}	1.450E 00	1.696E 02	4.436E 04	1.500E 07
T_1^{***}	2.460E -01	2.717E 01	4.827E 03	1.222E 06
T_2	4.738E 00	1.593E 03	5.995E 05	2.676E 08
T_2^*	2.794E 00	2.890E 02	7.187E 04	2.373E 07
T_2^{**}	7.416E -01	4.724E 01	8.169E 03	2.016E 06
T_2^{***}	5.357E -02	7.162E 00	8.796E 02	1.639E 05

	n=1	3	5	7
Δ	5.964E 00	1.884E 05	8.262E 09	3.744E 14
α_1	-1.105E 00	-1.933E -05	-8.599E -09	-1.168E -11
α_2	1.620E 00	6.188E -05	4.539E -08	8.642E -11

$s = .75\ n^2$

	n=1	3	5	7
T	7.117E -01	1.860E -02	9.417E -04	7.008E -05
$.5n\pi T^*$	-8.058E -01	-1.394E -02	-6.856E -04	-5.038E -05

$s = 2.5\ n^2$

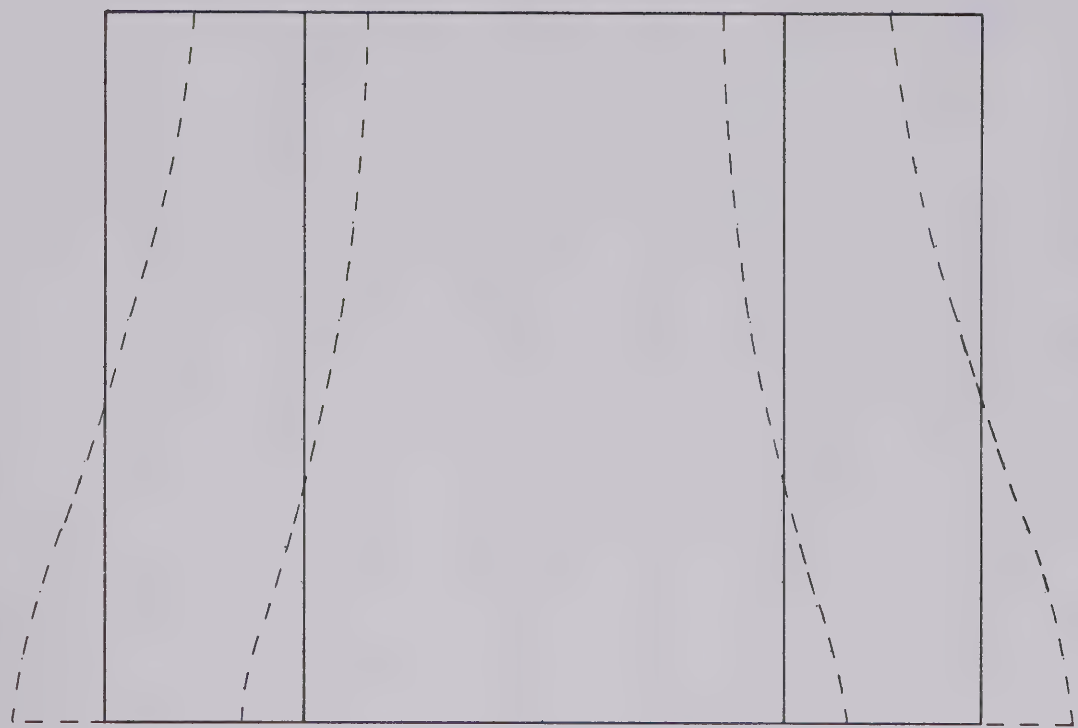
	n = 1	3	5	7
T	9.977E -01	7.143E -02	1.661E -02	6.181E -03
$.5n\pi T^*$	-1.445E -01	1.782E -02	5.011E -03	1.989E -03

$s = .75\ n^2$

	u	v
z=0	7.313E -01	0.0
.10	6.878E -01	-2.610E -01
.20	5.691E -01	-4.868E -01
.30	4.007E -01	-6.556E -01
.40	2.058E -01	-7.582E -01
.50	0.0	-7.925E -01
.60	-2.058E -01	-7.582E -01
.70	-4.007E -01	-6.556E -01
.80	-5.691E -01	-4.868E -01
.90	-6.878E -01	-2.610E -01
1.0	-7.313E -01	0.0

$s = 2.5\ n^2$

	u	v
z=0	1.092E 00	0.0
.10	9.872E -01	-2.360E -02
.20	7.666E -01	-6.985E -02
.30	5.244E -01	-1.158E -01
.40	2.621E -01	-1.467E -01
.50	0.0	-1.592E -01
.60	-2.621E -01	-1.467E -01
.70	-5.244E -01	-1.158E -01
.80	-7.666E -01	-6.985E -02
.90	-9.872E -01	-2.360E -02
1.0	-1.092E 00	0.0



PERTURBATION of a LAMINATED CYLINDER

$M_1 = +.2$

BIBLIOGRAPHY

[1] Ericksen, J.L., and Rivlin, R.S.; Large Elastic Deformations of Homogeneous Anisotropic Materials; Jour. Rat. Math. Mech., 3, p. 281-301, 1954.

[2] Green, A.E., and Zerna, W.; Theoretical Elasticity, 2nd Edition; Clarendon Press, Oxford, 1968.

[3] Rivlin, R.S.; The Fundamental Equations of Non-linear Continuum Mechanics; Dynamics of Fluids and Plasmas, Proceedings, p. 83-126, S.I. Pai, editor; Academic Press, New York, London, 1966.

[4] Truesdell, C.A.; The Elements of Continuum Mechanics; Springer-Verlag, New York, 1967.

[5] Vaughan, H.; The Finite Compression of Elastic Solid Cylinders in the Presence of Gravity; Proc. Roy. Soc. Lond. Ser. A, 321, p. 381-396, 1971.

[6] Wang, C.C.; On the Geometric Structures of Simple Bodies, a Mathematical Foundation for the Theory of Continuous Distributions of Dislocations; Arch. Rational Mech. Anal., 27, p. 33-94, 1967.

[7] Wang, C.C.; Universal Solutions for Incompressible Laminated Bodies; Arch. Rational Mech. Anal., 29, p. 161-192, 1968.